STOCHASTIC VOLATILITY AND MULTI-DIMENSIONAL MODELING IN THE EUROPEAN ENERGY MARKET

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Linda Vos, Oslo, June 2012

Preface

This thesis is submitted as partial fulfillment of the degree of *philosophiae doctor* (Ph.D.) at the university of Oslo. The work has been performed during my Ph.D. scholarship lasting from August 2008 to August 2011. Most of the work is carried out during my main period in the Center of Mathematics for Applications (CMA) at the University of Oslo under the guidance of my main supervisor Fred Espen Benth. However part of the work has been carried out during 2 visits, to the university of Agder, Kristiansand and the center of advanced studies at the Technical University of Munich. During this stays I got the opportunity to work with Steen Koekebakker (University of Agder) and Claudia Klüppelberg (Technical university of Munich).

This thesis consists of four papers, all of them are submitted. I wrote one of them alone and contributed substantially to the other ones.

Contents

Chapter 1

Introduction

Introduction

Since the 1 of July 2004 the energy market in Europe is officially liberalized. In the European Union the liberalization of power markets has been driven by the directive 96/92/EC of the European Parliament and Council. The directive is aimed at opening up the member states' electricity markets, in order to increase the number of producers and consumers which can negotiate the purchase and sale of electricity. Since then, a large number of electricity exchanges have opened in Europe. England and wales were the first to open an electricity exchange in 1990. After that Norway followed in 1993 with Statnett Marked (Nasdaq OMX commodities) [23]. The English and the Norwegian markets were already rather reasonably liberalized. The gas and electricity markets are owned by the private sector now in England and Norway. The markets in other countries like France, Italy, Netherlands, Finland and Sweden were owned by companies who had a government monopoly [17]. However, at the end of the 90-ties the Netherlands as well as Finland and Sweden got a liberalized market. The southern European countries took a bit more time before being liberalized.

The liberalization of energy markets gave rise to many regulation issues. Still regulations are changed in order to get a better functioning market. How to regulate the market is a separate field of studies, which will not be considered in this thesis. More interesting for a mathematician is that the energy exchanges give new financial data where lot of things are still unclear and interesting studies can be done. In order to model the energy market new theories can be developed. We will focus on the stochastic modeling of the electricity market.

1.1 Spot market

In this section we will give a short overview of common stochastic models used in order to model the electricity spot market. This will be used to put the papers of which this Ph.d. thesis consists in a context. For a more complete overview on energy modeling the author would like to refer to [3], [11], [12], [18] and [24].

The energy market distinguishes itself from other commodity markets by the non-storability of the products electricity and gas. This affects how the market is regulated and what kind of products are sold.

By the energy spot market actually the one-day ahead market is meant. Contracts are sold for delivering 1 Mega Watt electricity during one hour (1 MWh) for the different hours of the next day, as well as for MWh electricity for blocks of hours during the next day. In a sense these contracts are forward contracts, since the spot contracts are about electricity which will be delivered in the future (one-day head). The price of these one-day ahead contracts is determined by supply and demand during an auction at noon. Here the difference is with an ordinary forward contract, since that can be traded till delivery starts. The contracts that are sold on the one day-ahead market can be traded till delivery starts on the real-time market. Different real-time markets exists throughout Europe (for instance Elbas (Nordic countries), EEX intraday (Germany)).

What normally is denoted by the base spot price of an electricity market is an index, which is given by the average of all the hourly prices. Or in the case of the peak spot price the average of the hourly prices in the peak hours (between 8 a.m. and 8 p.m). The peak hours are the hours when there is naturally more demand for energy.

Remarkably is that the spot itself is not traded (it is an average of products which are traded). This leads to the possibility of structure in the spot price data. In a way the energy market is comparable to the interest rate market. Also the short-rate is not traded in itself. Moreover the economical relationship between the short rate and the bond-market is comparable to the relationship between the energy spot and the forward contracts. This is why many bond market models are used in the modeling of the energy market. However some adjustments have to be made. Typical features of the energy market are extreme spikes, seasonal behavior and stochastic volatility. These have to be accounted for in the models.

The spikes occur when there are misbalances between supply and demand of energy. Since energy can only be stored to a limited extend, the demand and supply have to match approximately. Both the demand and the supply sides are inelastic. On the supply side it is impossible to turn on a power plant on short notice. Depending on the kind of power plant this can take days (nuclear) or hours (gas, petrol). On the demand side the customer is not aware of the market structure. Most customers have a contract with an energy company and pay one pre-defined price for electricity. They are not aware that the price can change from hour to hour. Moreover electricity is often needed to keep a business running and it can be more expensive to slow down an industry then to pay more for electricity. This in-elasticity on both sides has lead to the possibility of enormous price spikes.

A second feature of the energy market is its seasonal behavior. In many countries there is naturally more demand for energy during the winter than during the summer, because of heating. Therefore the price of energy is often higher during winter than during summer. In some states with a warmer climate, like in California, it is the other way around. Here the demand for energy in order to heat during the winter is not so high. However during the summer air-conditioning demand, lots of energy. To capture this seasonality a seasonal function is introduced in energy models.

A last feature we will focus on in this thesis is stochastic volatility. Empirical studies by Trolle and Schwartz [20] confirmed stochastic volatility in energy markets. This means that the volatility is changing stochastically over time. Moreover there is evidence of a so-called inverse leverage effect. The volatility tends to increase with the level of power prices, because there is a negative relationship between inventory and prices (see for instance Deaton and Leroque [13]). In order to model stochastic volatility we have chosen to follow the approach of Barndorff-Nielsen and Shephard [1]. The stochastic volatility (SV) model of Barndorff-Nielsen and Shephard is statistically tractable and fits well on the existing literature of energy modeling. In Chapter 2

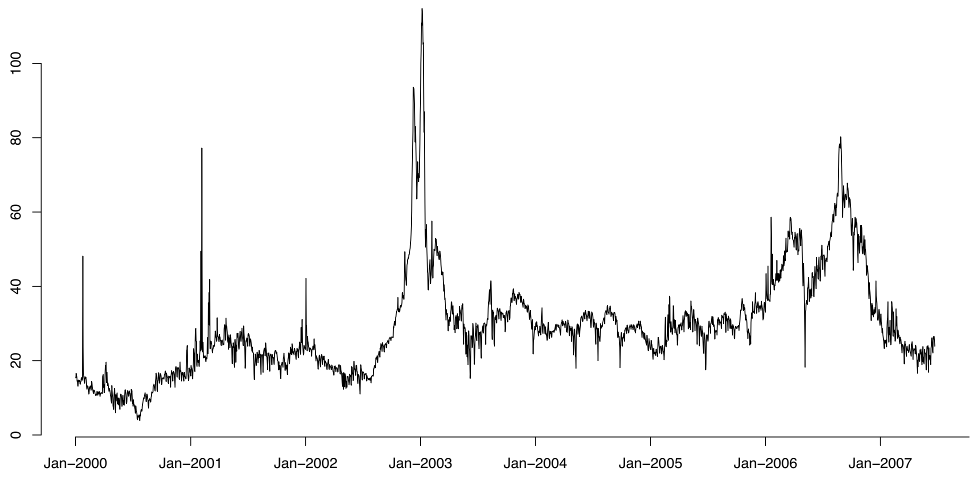

Figure 1.1: A typical electricity spot prices series. Depicted is Nordpool spot price data.

in this thesis the effect of stochastic volatility in pricing path dependent options is investigated. This is done using the Barndorff-Nielsen and Shephards stochastic volatility model (BNS SV model). Here ordinary stock data is used to estimate the parameters. However similar results are expected when energy data would have been used to estimate the parameters and instead of Asian options the valuation of an electricity line or gas pipe would have been considered.

One of the most common models is to model the spot as a combination of an Ornstein-Uhlenbeck processes and a seasonal function. This can either be done arithmetically or geometrically ([3]). A common geometric model is given by

$$S(t) = \Lambda(t) \cdot \exp(X(t) + \sum_i Y_i(t)) \tag{1.1.1}$$

here Λ is a seasonality function. The normal variation is given by X, an Ornstein-Uhlenbeck (OU) process with the following stochastic differential equation (SDE)

$$dX(t) = aX(t)\,dt + \sigma\,dW(t)$$

here $a < 0$ and $\sigma > 0$ are constants and W is a Brownian motion. The Y_i's are spike processes modeling the extreme spikes in the electricity market. The Y_i's are Ornstein-Uhlenbeck processes with the following SDE's.

$$dY_i(t) = b_i Y_i(t)\, dt + \eta_i\, dL_i(t)$$

here $b_i < 0$ and $\eta_i > 0$ are constants. The L_i's are Lévy processes possibly from different distributions.

Or similarly one can put the same variables in an arithmetic model

$$S(t) = \Lambda(t) + X(t) + \sum_i Y_i(t) \tag{1.1.2}$$

There is a discussion going on whether energy spot prices should be modeled geometrically or arithmetically. In financal modeling often geometric models are chosen (as this would be a natural choice). However, working with arithmetic models is computationally easier. Moreover in the energy market also negative prices are observed. For instance, the German power exchange EEX permitted negative price outcomes for the spot auction from autumn 2008. Since then negative prices have occurred. Sometimes it is more expensive to shut down a power plant then to pay somebody to use the electricity. These are of course rare situations. Negative prices can not be modeled by a geometric model. Therefor some argue that an arithmetic model should be preferred. Moreover Lucia and Schwartz [19] have found statistical evidence in the Nordpool market that arithmetic models do a better job in explaining prices then geometric models.

In chapter 3 and 5 in this thesis variations on this model are treated. In chapter 3 a multi-dimensional version of the geometric model (1.1.2) is introduced. Here, instead of static volatility, a stochastic volatility process is chosen. In order to model the normal variation X a multi-dimensional version of the BNS-SV model (Barndorff-Nielsen and Stelzer [2]) was taken. When no spike processes are added this model is a multivariate extension of the one-factor Schwartz model with stochastic volatility.

In chapter 5 the arithmetic model (1.1.1) is chosen with the possibility of a higher auto-regressive order. Instead of an OU-process a continuous auto-regressive moving average (CARMA) process (Brockwell [10]) is chosen. This is a continuous version of a auto-regressive moving average (ARMA) process. The OU-process is a special case of a CARMA process. Weron [24] and Pilopovic [18] have given a more extended overview of time-serie modeling in energy markets. As a driver of the CARMA process a stable process (Samorodnitsky and Taqqu [22]) is chosen. The Brownian motion is a special case of a stable process. Stable processes are known for their heavy tails and are therefore suitable to model extreme events like price spikes. When working with stable processes it is unnecessary to add several spike processes since one stable process can capture everything.

1.2 Forward market

Unlike more classical commodity markets like agriculture and metals, energy-related futures contracts deliver the underlying spot over a contracted period. There are for example contracts sold which deliver 1 MWh during every hour of the month in the future. Also year contracts and quarter of a year contracts are sold. In some markets also week contracts and 3 year contracts are sold. Contracts can be settled either physically or financially. Forwards are traded continuously till settlement starts.

Different to other commodity markets delivery is done over a period instead of at one point in time. This is due to the difficulties of energy storage. Mathematically this lead to some differences. From the theory of mathematical finance (Duffie [14]), we know that the value of any derivative is given as the present expected value of its payoff, where the expectation is taken with respect to a risk-neutral probability Q. It holds that

$$F(t,T_1,T_2) = \mathbb{E}_Q\left[\int_{T_1}^{T_2} w(u,T_1,T_2)S(u)du \,\middle|\, \mathcal{F}_t\right] \tag{1.2.1}$$

here Q is a risk neutral measure, t is the time, T_1 is the beginning of the delivering period and T_2 is the end of the delivering period. w is a weight function, which should be chosen according to the market structure. In electricity markets the swap price F is normally denoted by price per MWh. Therefore $w(u,T_1,T_2)$ is taken $\frac{1}{T_2-T_1}$. However when settlement is done continuously over the delivering period

$$w(u,T_1,T_2) = \frac{e^{-ru}}{\int_{T_1}^{T_2} e^{-rv}\,dv} \tag{1.2.2}$$

here r is the interest rate. This to adjust for the advantage of having money at the bank.

In geometric models it is often not possible to calculate the swap F explicitly using equation (1.2.1). However it is possible to given an expression in the form of an integral which can be approximated nummerically. From no arbitrage theory we know that if the forward would have been delivered at one point in time τ the forward price f would have been given by

$$f(t,\tau) = \mathbb{E}_Q[S(\tau)|\mathcal{F}_t] \tag{1.2.3}$$

meaning that the forward price is the best risk-neutral prediction at time t of the spot price $S(\tau)$ at delivery. The swap F may be viewed as a continuous flow of forwards f (see Prop. 4.1 in Benth, Saltyte Benth and Koekebakker [3]).

$$F(t,T_1,T_2) = \int_{T_1}^{T_2} w(\tau,T_1,T_2)f(t,\tau)\,d\tau \tag{1.2.4}$$

In chapter 4 the derivation (1.2.3) is done for the multivariate volatility model introduced in Benth and Vos [7]. Moreover some properties of the derived forward curves are analyzed. Examples of possible shapes of the forward curve using this model are given and transform based methods to price options on the forward curves are given. In chapter 5 the derivation (1.2.1) is done for the CARMA model introduced in García, Klüppelberg and Müller [15]. Furthermore using parameters estimated on the spot price the risk premium is empirically analyzed.

The risk premium is defined as the difference between the futures price and the predicted spot, that is, in terms of electricity futures contracts,

$$R_{pr}(t,T_1,T_2) = F(t,T_1,T_2) - \mathbb{E}\left[\int_{T_1}^{T_2} w(\tau,T_1,T_2)\, S(\tau)\, d\tau \mid \mathcal{F}_t\right]. \tag{1.2.5}$$

The risk premium can be used to identify the risk-measure Q. In chapter 5 is found that the risk premium is negative in the long end of the market and positive in the short end of the market. The positive risk premium for contracts close to delivery tells us that the demand side (retailers and consumers) of the market is willing to pay a premium for locking in electricity prices as a hedge against spike risk (see Geman and Vasicek [16]). In the long end of the market the risk premium is negative meaning that on the supply side the energy companies want to hedge their risk and willing to accept a lower price. The energy companies hedge the risk of an uncertain spot price. These empirical findings are in line with results of empirical studies done by Benth, Cartea and Kiesel [4].

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Chapter 2

Path dependent options and the effect of stochastic volatility

LINDA VOS

Chapter 3

Cross-commodity spot price modeling with stochastic volatility and leverage for energy markets

FRED ESPEN BENTH AND LINDA VOS

Abstract

Spot prices in energy markets exhibit special features like price spikes, mean-reversion, stochastic volatility, inverse leverage effect and dependencies between the commodities. In this paper a multivariate stochastic volatility model is introduced which captures these features. The second order structure and stationarity of the model are analysed in detail. A simulation method for Monte Carlo generation of price paths is introduced and a numerical example is presented.

1 Introduction

Energy markets world-wide have been liberalized over the last decades to create liquid trading arenas for power commodities like electricity, gas, and coal. The markets are continuously developing, and in recent years gradually becoming more and more connected. For instance, interconnectors between UK, Scandinavia and continental Europe integrate the power markets. Also, different electricity markets on the continental Europe exchange to a large extent energy across borders. As a reflection of this market integration is the growing need for multivariate price models for power. This includes cross-commodity models for gas and electricity, say, but also models for electricity traded in different but integrated markets. In this paper we propose and analyse a multivariate spot price model for power.

Power market spot prices have several distinct characteristics. Typically, spot prices spike occationally when there is an imbalance in supply and demand, since the supply curve is inelastic. Further, the market prices are moving with the season, with high prices in winter due to heating, or in summer due to air-conditioning cooling. Prices also naturally mean revert due to demand and supply forces. Partly because of the spikes, the prices observed in gas and electricity markets are to a large extent leptokurtic. In fact, volatility may easily reach above 100%. A discussion of the features of power spot prices can be found in Eydeland and Wolyniec [16] and Geman [17]. There exists many models for spot price dynamics in power markets, and we refer to Benth *et al.* [9] for an overview and analysis.

In energy markets there is evidence of a so-called inverse leverage effect. The volatility tends to increase with the level of power prices, since there is a negative relationship between inventory and prices (see for instance Deaton and Leroque [15]). Little available inventory means higher and more volatile prices. This is reflected in gas markets where storage facilities play an important role in price determination. There is also evidence for dependence between different commodities. For instance, it is unlikely that the price of gas and electricity in the UK market, say, will drive too far apart, since gas is the dominating fuel for power production. Likewise, since gas can be transported as liquid natural gas (LNG), different gas markets can not have prices which become increasingly different.

In recent years there has been an interest in stochastic volalility models for commodities, and in particular energy. In Hikspoors and Jaimungal [18] we find an analysis of forward pricing in commodity markets in the presence of stochastic volatility. Several popular models are treated. More recently, Trolle and Schwartz [30] introduced the notion of unspanned volatility, and analysed this in power markets. Their statistical analysis confirms the presence of stochastic volatility in commodity markets. Benth [7] applied the Barndorff-Nielsen and Shepard stochastic volatility model in commodity markets, and derived forward prices based on this. An empirical study on UK gas prices was performed.

In this paper we propose a stochastic dynamic for cross-commodity spot price modelling generalizing the univariate dynamics studied in Benth [7]. The model is flexible enough to capture spikes, mean-reversion and stochastic volatility. Moreover, it includes the possibility to model inverse leverage. Our proposed dynamics can model co- and independent jump behaviour (spikes) in cross-commodity markets. Also, the model allows for analytical forward prices. This issue, along with pricing of derivatives on spots and forwards, are left to the follow-up paper by Benth and Vos [12].

The spot price dynamics we propose are based on Ornstein-Uhlenbeck processes driven by multivariate subordinators. The logarithmic price dynamics are defined by multi-factor processes and seasonal functions to account for deterministic variability over a year. The stochastic volatility processes are multi-variate as well, so that we can incorporate second-order dependencies between commodities. The volaltity model is adopted from the so-called Barndorff-Nielsen and Shephard model (BNS for short), extended to a multivariate setting (see Barndorff-Nielsen and Shephard [4] and Barndorff-Nielsen and Stelzer [6]). As for the multi-dimensional extension, the volatility is modeled with a matrix-valued Ornstein-Uhlenbeck process driven by a positive definite matrix-valued subordinator (see Barndorff-Nielsen and Pérez-Abreu [3]). We prove that the spot prices are stationary, and compute the characteristic function of the stationary distribution. Several other probabilistic features of the model are presented and discussed, demonstrating its flexibility in modelling prices and its analytical tractability. From a more practical point of view, a method for simulating the prices is presented. We provide an empirical example where the algorithm is applied. Our approach is influenced by the work of Stelzer [29].

The paper is organized as follows. Section 2 introduces the spot model, thereafter the stationary distribution and the probabilistic properties of the various factors of the model are deduced in Section 3. The following section deals with the same properties of the spot price model. Section 5 gives an empirical example and a method to perform Monte-Carlo simulation of the model. Finally, in Section 6 we conclude.

Notation

For the convenience of the reader, we have collected some frequently used notations. We adopt the notation used by Pirgorsch and Stelzer [22]. Throughout this paper we write $\mathbb{R}_+$ for the positive real numbers and we denote the set of real $n \times n$ matrices by $M_n(\mathbb{R})$. We denote the group of invertible matrices by $GL_n(\mathbb{R})$, the linear subspace of symmetric matrices by $\mathbb{S}_n$, the positive definite cone of symmetric matrices by $\mathbb{S}_n^+$. I_n stands for the $n \times n$ identity matrix, $J_n(v)$ is an operator $\mathbb{R}^n \to M_n(\mathbb{R})$ which creates a diagonal matrix with the vector $v \in \mathbb{R}^n$ on the diagonal, $diag(A)$ is a vector in $\mathbb{R}^n$ consisting of the diagonal of the matrix $A \in M_n(\mathbb{R})$, $\sigma(A)$ denotes the spectrum (the set of all eigenvalues) of a matrix $A \in M_n(\mathbb{R})$. The tensor (Kronecker) product of two matrices A, B is written as $A \otimes B$. vec denotes the well-known vectorization operator that maps the $n \times n$ matrices to $\mathbb{R}^{n^2}$ by stacking the columns of the matrices below one another. Furthermore, we denote $tr(A)$ for the trace of the matrix $A \in M_n(\mathbb{R})$, which is the sum of the elements on the diagonal. The transpose of the matrix $A \in M_n(\mathbb{R})$ is denoted A^T while A_{ij} is the element of A in the i-th row and j-th column. The unit vector with on the i-th place a one is denoted e_i. For $A \in M_n(\mathbb{R})$, we denote the operator $\mathbf{A}$ associated with the matrix A as $\mathbf{A} : X \mapsto AX + XA^T$. This operator can be represented as $\mathrm{vec}^{-1} \circ ((A \otimes I_n) + (I_n \otimes A)) \circ \mathrm{vec}$. Its inverse is denoted by $\mathbf{A}^{-1}$, which exists whenever $I \otimes A + A \otimes I$ is invertible. In this case, we can represent $\mathbf{A}^{-1}$ by $\mathrm{vec}^{-1} \circ ((A \otimes I_n) + (I_n \otimes A))^{-1} \circ \mathrm{vec}$. Remark that $A \otimes I_n + I_n \otimes A$ is equal to the Kronecker sum of the matrix A with itself.

2 The cross-commodity spot price model

Suppose we are given a complete filtered probability space $(\Omega, \mathcal{F}, P)$ equipped with the filtration $\{\mathcal{F}_t\}_{t \geq 0}$ satisfying the usual conditions (see e.g. Protter [24]). Assume $m, n \in \mathbb{N}$ with $0 \leq m < n$. Let $\{\widetilde{L}_j(t)\}_{t \in \mathbb{R}^+} \in \mathbb{S}_d^+$, $j = 1, \ldots, n$ be n independent matrix-valued subordinators as introduced in Barndorff-Nielsen and Pérez-Abreu [3]. Furthermore, let L_i, $i = 1, \ldots, m$ be $\mathbb{R}^d$-valued subordinators[1]. For $i = 1, \ldots, m$ the vector-valued subordinators L_i are formed by taking the diagonal of the matrix-valued subordinators $\widetilde{L}_i(t)$. This implies that L_i will jump whenever $\widetilde{L}_i$ does. If one of the off-diagonal elements jumps, also the diagonal element has to jump in order to keep the volatility process $\widetilde{L}_i$ in the positive definite cone $\mathbb{S}_d^+$. The subordinators are assumed to be driftless, and we choose to work with the versions which are right-continuous with left limits (RCLL, for short). Moreover, let W be a standard $\mathbb{R}^d$-valued Brownian motion independent of the subordinators.

We define the spot price dynamics of d commodities as follows: let

$$S(t) = \Lambda(t) \cdot \exp\left(X(t) + \sum_{i=1}^{m} Y_i(t)\right), \tag{2.1}$$

where $\Lambda : [0, T] \mapsto \mathbb{R}_+^d$ is a vector of bounded measurable seasonality functions, '$\cdot$' denotes coordinate-wise multiplication, and

$$dX(t) = AX(t)\, dt + \Sigma(t)^{1/2}\, dW(t)\,, \tag{2.2}$$

[1] A multivariate subordinator is a Lévy process which is increasing in each of its coordinates (see Sato [1]).

$$dY_i(t) = (\mu_i + B_i Y_i(t))\ dt + \eta_i\, dL_i(t)\,, \tag{2.3}$$

for $i = 1, \ldots, m$. A, B_i's and η_i are in $GL_d(\mathbb{R})$ and μ_i is a vector in $\mathbb{R}^d$. To ensure the existence of stationary solutions we assume that the eigenvalues of the matrices A, B_i have negative real-parts. In order to have the Itô integral in (2.2) well-defined, we suppose that

$$P\left(\int_0^T tr(\Sigma(t))\, dt < \infty\right) = 1\,. \tag{2.4}$$

Here, $T < \infty$ is some terminal time for our energy markets. The entries of η_i can be negative. So although L_i is a $\mathbb{R}^d$-valued subordinator, there can be negative jumps in the spot-price process.

The stochastic volatility process $\Sigma(t)$ is a superposition of positive-definite matrix valued Ornstein-Uhlenbeck processes as introduced in Barndorff-Nielsen and Stelzer [6],

$$\Sigma(t) = \sum_{j=1}^{n} \omega_j Z_j(t)\,, \tag{2.5}$$

with

$$dZ_j(t) = (C_j Z_j(t) + Z_j(t) C_j^T)dt + d\widetilde{L}_j(t)\,, \tag{2.6}$$

and the ω_j's are weights summing up to 1. Moreover, $\{C_j\}_{1\leq j\leq n} \in GL_d(\mathbb{R})$. To ensure a stationary solution we will assume that the eigenvalues of C_j have negative real-parts. This stochastic volatility model is a multivariate extension of the so called BNS SV model introduced by Barndorff-Nielsen and Shephard [4] for general asset price processes. The commodity spot price dynamics with the BNS SV model as stochastic volatility structure is a generalization of the univariate spot model analysed in Benth [7].

Note that Y_i and Σ_i for $i = 1, \ldots, m$ have related subordinators L and $\widetilde{L}$ driving the noise. Thus, when the volatility component Σ jumps, we observe simultaneously a change in the spot price. Hence, we can have an inverse leverage effect since increasing prices follow from increasing volatility, and vice versa (see Eydeland and Wolyniec [16] and Geman [17] for a discussion on inverse leverage in power markets). We also have $n - m$ independent stochastic volatility components Z_j, $j = m + 1, \ldots, n$ that do not directly influence the price process paths but have a second order effect. The processes Y_i can be interpreted as modeling the spikes, whereas X is the normal variation in the market. The latter is also sometimes referred to as the base component of the price variations.

By turning off the processes Y_i (choose $\mu_i = \eta_i = 0$ and $B_i = 0$ for all i), we obtain a multivariate extension of the Schwartz model with stochastic volatility and stock-price dynamics:

$$S(t) = \Lambda(t) \cdot \exp(X(t)) \tag{2.7}$$

where $X(t)$ is defined in (2.2). The Schwartz model with constant volatility is a mean-reversion process proposed by Schwartz [28] for spot price dynamics in commodity markets like oil. In order to have spikes being independent of the volatility process $\Sigma(t)$, we can switch off some of the ω_j's in (2.5), that is choose some $\omega_j = 0$. Then the L_i's from the corresponding $\widetilde{L}_j$'s will give rise to independent spike components.

In electricity markets one observes spikes in the spot price dynamics (see e.g. Benth *et. al.* [9]). These spikes often occur seasonally. In the Nordic electricity market Nord-Pool, price spikes occur in the winter time when demand is high. We therefore may wish the jump frequency of the subordinators L_i, $i = 1, \ldots, m$ to be time-dependent. This is not possible when working with Lévy processes, but we can generalize to independent increment processes instead (see Jacod and Shiryaev [20]). Independent increment processes can be thought of as *time-inhomogeneous* Lévy processes. Our modeling and analysis to come are easily modified to include such processes. To keep matters slightly more simplified, we stick to the time-homogeneous case here. The interested reader is referred to Benth *et al* [9] for applications of independent increment processes in energy markets.

We assume the following integrability conditions for the subordinators.

$$\mathbb{E}\left[\log^+ \|\widetilde{L}_j(1)\|\right] < \infty\,, \tag{2.8}$$

where $\log^+(x)$ is defined as $\max(\log(x), 0)$ and $j = 1, \ldots, n$ and $\|A\|^2 = tr(A^T A)$ is the Frobenius norm of the matrix $A \in M_d(\mathbb{R})$. Note that this condition implies

$$\mathbb{E}\left[\log^+ |L_i(1)|\right] < \infty\,, \tag{2.9}$$

for $i = 1, \ldots, m$ and $| \cdot |$ is the Euclidean 2-norm on $\mathbb{R}^d$.

In the next Section we study the probabilistic properties of the factor processes X and Y_i. As we shall see, the analysis of the spot price model will depend crucially on the properties of certain operators, which will reflect back to restrictions on the matrices A, B_i and C_j, for $i = 1, \ldots, m$ and $j = 1, \ldots, n$. Throughout the rest of the paper, we suppose that $\mathbf{A}$, $\mathbf{B}_i$ and $\mathbf{C}_j$ are invertible for $i = 1, \ldots, m$ and $j = 1, \ldots, n$. Furthermore, the matrices A and C_j are commuting, for each $j = 1, \ldots, n$. Finally, the operators $\mathbf{A} - \mathbf{C}_j$ are invertible for $j = 1, \ldots, n$.

3 Stationarity and probabilistic properties of the factor processes

The processes X, Y_i are Ornstein-Uhlenbeck processes. Applying the multi-dimensional Itô formula (see Ikeda and Watanabe [19]) to the stochastic differential equations yields the following solutions: for $0 \leq s \leq t$,

$$X(t) = \mathbf{e}^{A(t-s)} X(s) + \int_s^t \mathbf{e}^{A(t-u)} \Sigma(u)^{1/2} \, dW(u) \,, \tag{3.1}$$

$$Y_i(t) = \mathbf{e}^{B_i(t-s)} Y_i(s) + B_i^{-1}(I - \mathbf{e}^{B_i(t-s)})\mu_i + \int_s^t \mathbf{e}^{B_i(t-u)} \eta_i \, dL_i(u) \,, \tag{3.2}$$

for $i = 1, \ldots, m$. The matrix exponentials are defined as usual as $\mathbf{e}^A := I + \sum_{i=1}^{\infty} \frac{A^n}{n!}$.

According to Barndorff-Nielsen and Stelzer [6], Sect. 4, the solution of $Z_j(t)$, $j = 1, \ldots, n$, is given by

$$Z_j(t) = \mathbf{e}^{C_j(t-s)} Z_j(s) \mathbf{e}^{C_j^T(t-s)} + \int_s^t \mathbf{e}^{C_j(t-u)} \, d\widetilde{L}_j(u) \mathbf{e}^{C_j^T(t-u)} \,. \tag{3.3}$$

The matrix-valued stochastic integral in the second term of $Z_j(t)$ is understood as follows. For two $M_d(\mathbb{R})$-valued bounded and measurable functions $E(u)$ and $F(u)$ on $[t, s]$, the notation $\int_s^t E(u) \, d\widetilde{L}(u) F(u)$ means the matrix $G(s, t) \in M_d(\mathbb{R})$ with coordinates defined by

$$G_{ij}(s, t) = \sum_{k=1}^{d} \sum_{l=1}^{d} \int_s^t E_{ik}(u) F_{lj}(u) \, d\widetilde{L}_{kl}(u) \,.$$

Here, $\widetilde{L}$ is the generic notation for some $\widetilde{L}_j$. We remark that since $\widetilde{L}_j$ are supposed to be RCLL, the processes Z_j also are RCLL.

Let us first look at the expected values of X and Y_i. For this, the following Lemma, which is interesting in its own right, is useful:

Lemma 3.1. *Let L be an integrable Lévy process in $\mathbb{R}^d$and f a bounded measurable function from $[s, t]$ into $M_d(\mathbb{R})$ being of bounded variation. Then it holds that*

$$\mathbb{E}\left[\int_s^t f(u) dL(u)\right] = \int_s^t f(u) du \, \mathbb{E}[L(1)] \,. \tag{3.4}$$

Proof. Define the Lévy process $\widehat{L}(u) := L(u) - \mathbb{E}[L(1)]u$, which has expectation zero. From integration by parts (use the multi-dimensional Itô Formula for jump processes in Ikeda and Watanabe [19]), it holds

$$\int_s^t f(u) \, d\widehat{L}(u) = f(t)\widehat{L}(t) - f(s)\widehat{L}(s) - \int_s^t \widehat{L}(u) \, df(u) \,.$$

Now, choosing the right-continuous with left limits version of L (as we always can do for Lévy processes), we can apply the Fubini-Tonelli Theorem to conclude that

$$\mathbb{E}\left[\int_s^t f(u)\, d\widehat{L}(u)\right] = 0\,,$$

and hence the Lemma follows. □

We find the following conditional expectations for the factor processes:

Lemma 3.2. *Suppose that $L_i(1)$ are integrable for $i = 1, \ldots, m$. Then it holds*

$$\begin{aligned}
&\mathbb{E}[X(t)|\mathcal{F}_s] = \boldsymbol{e}^{A(t-s)}X(s)\,,\\
&\mathbb{E}[Y_i(t)|\mathcal{F}_s] = \boldsymbol{e}^{B_i(t-s)}Y_i(s) + B_i^{-1}(I - \boldsymbol{e}^{B_i(t-s)})\mu_i + B_i^{-1}\left(\eta_i - \boldsymbol{e}^{B_i(t-s)}\eta_i\right)\mathbb{E}\left[L_i(1)\right]\,,
\end{aligned}$$

for $i = 1, \ldots, m$

Proof. The conditional expectation of $X(t)$ is given by

$$\begin{aligned}
\mathbb{E}[X(t)|\mathcal{F}_s] &= \mathbf{e}^{A(t-s)}X(s) + \mathbb{E}\left[\int_s^t \mathbf{e}^{A(t-u)}\Sigma(u)^{1/2}\, dW(u)\right]\,,\\
&= \mathbf{e}^{A(t-s)}X(s) + \mathbb{E}\left[\mathbb{E}\left[\int_s^t \mathbf{e}^{A(s-u)}\Sigma(u)^{1/2}\, dW(u)|\Sigma(u)_{s\le u\le t}\right]\right]\,,\\
&= \mathbf{e}^{A(t-s)}X(s)\,.
\end{aligned}$$

In the third equality we use that the paths of $\Sigma(u)$ are right-continuous with left limits, and therefore bounded on $[s,t]$, and hence $u \mapsto \exp(A(s-u))\Sigma^{1/2}(s-u)$ is Itô integrable on $[t,s]$ in a strong sense. We can thus conclude that the expectation is zero of this Itô integral.

For Y_i, $i = 1, \ldots, m$, we get

$$\begin{aligned}
\mathbb{E}[Y_i(t)|\mathcal{F}_s] &= \mathbf{e}^{B_i(t-s)}Y_i(s) + B_i^{-1}(I - \mathbf{e}^{B_i(t-s)})\mu_i + \mathbb{E}\left[\int_s^t \mathbf{e}^{B_i(t-u)}\eta_i\, dL_i(u)|\mathcal{F}_s\right]\,,\\
&= \mathbf{e}^{B_i(t-s)}Y_i(s) + B_i^{-1}(I - \mathbf{e}^{B_i(t-s)})\mu_i + \int_s^t \mathbf{e}^{B_i(t-u)}\eta_i\, du \cdot \mathbb{E}\left[L_i(1)\right]\,,\\
&= \mathbf{e}^{B_i(t-s)}Y_i(s) + B_i^{-1}(I - \mathbf{e}^{B_i(t-s)})\mu_i + B_i^{-1}\left(\eta_i - \mathbf{e}^{B_i(t-s)}\eta_i\right)\mathbb{E}\left[L_i(1)\right]\,.
\end{aligned}$$

where we used Lemma 3.1 to obtain the last equality. □

Since A and B_i have eigenvalues with a negative real part, letting t tend to infinity yields

$$\lim_{t\to\infty}\mathbb{E}[X(t)\,|\,\mathcal{F}_s] = 0\,,$$

$$\lim_{t\to\infty} \mathbb{E}[Y_i(t)\,|\,\mathcal{F}_s] = B_i^{-1}\left(\mu_i + \eta_i \mathbb{E}[L_i(1)]\right)\,,$$

for $i = 1, \ldots, m$. Hence, in stationarity, the "base-term" $X(t)$ will contribute zero in expectation, whereas the "leverage-terms" Y_i will give a drift imposed from the subordinators and the coefficients μ_i.

Let us analyse the second-order properties of the factor processes. We have the following result for the variance of the "base component" $X(t)$:

Lemma 3.3. *Assume that $\widetilde{L}_j(1)$ is integrable for $j = 1, \ldots, n$. Then it holds*

$$\begin{aligned}
\mathrm{Var}[X(t)|\mathcal{F}_s] = &\sum_{j=1}^n \omega_j (\mathbf{A}-\mathbf{C}_j)^{-1}\left\{\boldsymbol{e}^{A(t-s)}Z_j(s)\boldsymbol{e}^{A^T(t-s)} - \boldsymbol{e}^{C_j(t-s)}Z_j(s)\boldsymbol{e}^{C_j(t-s)}\right\}\\
&+\sum_{j=1}^n \omega_j \mathbf{C}_j^{-1}\left\{(\mathbf{A}-\mathbf{C}_j)^{-1}\left\{\boldsymbol{e}^{A(t-s)}\mathbb{E}[\widetilde{L}_j(1)]\boldsymbol{e}^{A^T(t-s)} - \boldsymbol{e}^{C_j(t-s)}\mathbb{E}[\widetilde{L}_j(1)]\boldsymbol{e}^{C_j^T(t-s)}\right\}\right\}\\
&-\sum_{j=1}^n \omega_j \mathbf{A}^{-1}\left\{\mathbf{C}_j^{-1}\left\{\boldsymbol{e}^{A(t-s)}\mathbb{E}[\widetilde{L}_j(1)]\boldsymbol{e}^{A^T(t-s)} - \mathbb{E}[\widetilde{L}_j(1)]\right\}\right\}\,,
\end{aligned}$$

for $0 \le s \le t$.

Proof. We compute the conditional variance for the process X by appealing to the tower property of conditional expectations and the independent increment property of Brownian motion. Letting $\mathcal{G}_{s,t}$ be the σ-algebra generated by $\mathcal{F}_s$ and the paths $\Sigma(u)$, $s \le u \le t$, we find,

$$\begin{aligned}
\mathrm{Var}[X(t)|\mathcal{F}_s] &= \mathbb{E}\left[\left(\mathbf{e}^{A(t-s)}X(s) + \int_s^t \mathbf{e}^{A(s-u)}\Sigma(u)^{1/2}\,dW(u)\right)^2 |\mathcal{F}_s\right] - \mathbb{E}[X(t)|\mathcal{F}_s]^2\,,\\
&= \mathbb{E}\left[\mathbb{E}\left[\left(\int_s^t \mathbf{e}^{A(s-u)}\Sigma(u)^{1/2}\,dW(u)\right)^2 |\mathcal{G}_{s,t}\right]\, |\mathcal{F}_s\right]\,,\\
&= \mathbb{E}\left[\int_s^t \mathbf{e}^{A(t-u)}\Sigma(u)\mathbf{e}^{A^T(t-u)}du|\mathcal{F}_s\right]\\
&= \sum_{j=1}^n \omega_j \int_s^t \mathbf{e}^{A(t-u)}\mathbb{E}\left[Z_j(u)|\mathcal{F}_s\right]\mathbf{e}^{A^T(t-u)}\,du\,,
\end{aligned}$$

after appealing to Fubini's Theorem. From the explicit representation of $Z_j(t)$ in (3.3), we find

$$\begin{aligned}
\mathbb{E}[Z_j(u)|\mathcal{F}_s] &= \mathbf{e}^{C_j(u-s)}Z_j(s)\mathbf{e}^{C_j^T(u-s)} + \int_s^u \mathbf{e}^{C_j(u-v)}\mathbb{E}[\widetilde{L}_j(1)]\mathbf{e}^{C_j^T(u-v)}\,dv\\
&= \mathbf{e}^{C_j(u-s)}Z_j(s)\mathbf{e}^{C_j^T(u-s)} + \int_0^{u-s} \mathbf{e}^{C_j z}\mathbb{E}[\widetilde{L}_j(1)]\mathbf{e}^{C_j^T z}\,dz\\
&= \mathbf{e}^{C_j(u-s)}Z_j(s)\mathbf{e}^{C_j^T(u-s)} + \mathbf{C}_j^{-1}\left\{\mathbf{e}^{C_j(t-s)}\mathbb{E}[\widetilde{L}_j(1)]\mathbf{e}^{C_j^T(t-s)} - \mathbb{E}[\widetilde{L}_j(1)]\right\}\,,
\end{aligned}$$

after appealing to Lemma 3.1. Hence, using that A and C_j are commuting for each $j = 1, \dots, n$, we find

$$\begin{aligned}
\mathrm{Var}[X(t)|\mathcal{F}_s] &= \sum_{j=1}^{n} \omega_j \int_s^t \mathbf{e}^{A(t-u)} \mathbf{e}^{C_j(u-s)} Z_j(s) \mathbf{e}^{C_j^T(u-s)} \mathbf{e}^{A^T(t-u)}\, du \\
&\quad + \sum_{j=1}^{n} \omega_j \mathbf{C}_j \left\{ \int_s^t \mathbf{e}^{A(t-u)} \mathbf{e}^{C_j(u-s)} \mathbb{E}[\widetilde{L}_j(1)] \mathbf{e}^{C_j^T(u-s)} \mathbf{e}^{A^T(t-u)}\, du \right\} \\
&\quad - \sum_{j=1} \omega_j \mathbf{C}_j^{-1} \int_s^t \mathbf{e}^{A(t-u)} \mathbb{E}[\widetilde{L}_j(1)] \mathbf{e}^{A^T(t-u)}\, du \\
&= \sum_{j=1}^{n} \omega_j (\mathbf{A} - \mathbf{C}_j)^{-1} \left\{ \mathbf{e}^{A(t-s)} Z_j(s) \mathbf{e}^{A^T(t-s)} - \mathbf{e}^{C_j(t-s)} Z_j(s) \mathbf{e}^{C_j(t-s)} \right\} \\
&\quad + \sum_{j=1}^{n} \omega_j \mathbf{C}_j^{-1} \left\{ (\mathbf{A} - \mathbf{C}_j)^{-1} \left\{ \mathbf{e}^{A(t-s)} \mathbb{E}[\widetilde{L}_j(1)] \mathbf{e}^{A^T(t-s)} - \mathbf{e}^{C_j(t-s)} \mathbb{E}[\widetilde{L}_j(1)] \mathbf{e}^{C_j^T(t-s)} \right\} \right\} \\
&\quad - \sum_{j=1}^{n} \omega_j \mathbf{A}^{-1} \left\{ \mathbf{C}_j^{-1} \left\{ \mathbf{e}^{A(t-s)} \mathbb{E}[\widetilde{L}_j(1)] \mathbf{e}^{A^T(t-s)} - \mathbb{E}[\widetilde{L}_j(1)] \right\} \right\} .
\end{aligned}$$

The Lemma follows. □

Note that the explicit expression for the variance of the base component is computed under the condition of the matrices A and C_j being commutable. Moreover, we observe that for the Lemma to hold, we must have the imposed conditions of invertibility of the operators $\mathbf{A}$, $\mathbf{C}_j$ and $\mathbf{A} - \mathbf{C}_j$. Recalling that the matrices A and C_j have eigenvalues with negative real part, we pass to the limit $t \to \infty$ to find

$$\lim_{t\to\infty} \mathrm{Var}[X(t)] = \sum_{j=1}^{n} \omega_j \mathbf{A}^{-1} \mathbf{C_j}^{-1} \mathbb{E}[\widetilde{L}_j(1)] .$$

Observe that the stationary limit of the variance depends explicitly on the mean-reversion coefficient matrices A and C_j. In fact, from Barndorff-Nielsen and Stelzer [6] we know that the stationary expected value of $Z_j(s)$ is $\mathbf{C_j}^{-1}\mathbb{E}[\widetilde{L}_j(1)]$, so we can write

$$\lim_{t\to\infty} \mathrm{Var}[X(t)|\mathcal{F}_s] = \mathbf{A}^{-1} \lim_{t\to\infty} \mathbb{E}[\Sigma(t)] . \tag{3.5}$$

for the stationary variance of the base component.

We move on and find the variance of $Y_i(t)$:

Lemma 3.4. *Suppose that $L_i(1)$ are square integrable for $i = 1, \dots, m$. Then it holds,*

$$\mathrm{Var}[Y_i(t)|\mathcal{F}_s] = \mathbf{B_i}^{-1} \left(\eta_i \mathbb{E}[L_i(1) L_i^T(1)] \eta_i^T - \boldsymbol{e}^{B_i(t-s)} \eta_i \mathbb{E}[L_i(1) L_i^T(1)] \eta_i^T \boldsymbol{e}^{B_i^T(t-s)} \right)$$

$$- B_i^{-1}(I - \boldsymbol{e}^{B_i(t-s)})\eta_i \mathbb{E}[L_i(1)]\mathbb{E}[L_i^T(1)]\eta_i^T(I - \boldsymbol{e}^{B_i^T(t-s)})B_i^{-T} \,,$$

for $i = 1, \ldots, m$ *and* $0 \leq s \leq t$.

Proof. Fix a $i = 1, \ldots, n$. By (3.2), we find that the conditional variance of $Y_i(t)$ given $\mathcal{F}_s$ is

$$\mathrm{Var}[Y_i(t)|\mathcal{F}_s] = \mathrm{Var}[\int_s^t \mathbf{e}^{B_i(t-u)}\eta_i \, dL_i(u)|\mathcal{F}_s] \,.$$

Moreover, by the independent increment property of Lévy processes it holds

$$\mathrm{Var}[Y_i(t)|\mathcal{F}_s] = \mathrm{Var}[\int_s^t \mathbf{e}^{B_i(t-u)}\eta_i \, dL_i(u)] \,.$$

But, by Itô isometry for Lévy process integrals

$$\begin{aligned}
\mathbb{E}&\left[\int_s^t \mathbf{e}^{B_i(t-u)}\eta_i \, dL_i(u) \int_s^t \mathbf{e}^{B_i(t-u)}\eta_i \, dL_i(u)^T\right] \\
&= \int_s^t \mathbf{e}^{B_i(t-u)}\eta_i \mathbb{E}[L_i(1)L_i^T(1)]\eta_i^T \mathbf{e}^{B_i^T(t-u)} \, du \\
&= \mathbf{B}^{-1}\left(\eta_i \mathbb{E}[L_i 1(1)L_i^T(1)]\eta_i^T - \mathbf{e}^{B_i(t-s)}\eta_i \mathbb{E}[L_i(1)L_i^T(1)]\eta_i^T \mathbf{e}^{B_i^T(t-s)}\right)
\end{aligned}$$

On the other hand, following from Lemma 3.1

$$\begin{aligned}
\mathbb{E}\left[\int_s^t \mathbf{e}^{B_i(t-u)}\eta_i \, dL_i(u)\right] &= \int_s^t \mathbf{e}^{B_i(t-u)} \, du \eta_i \mathbb{E}[L_i(1)] \\
&= B_i^{-1}(I - \mathbf{e}^{B_i(t-s)})\eta_i \mathbb{E}[L_i(1)] \,.
\end{aligned}$$

Hence, the Lemma follows. □

Note that we have used the standing condition of invertibility of the operators $\mathbf{B}_i$ in this Lemma. We can also for $Y_i(t)$ compute an explicit stationary limit for the variance using that the eigenvalues of B_i have negative real parts:

$$\lim_{t\to\infty} \mathrm{Var}[Y_i(t)|\mathcal{F}_s] = \mathbf{B_i}^{-1}\eta_i \mathbb{E}[L_i(1)L_i^T(1)]\eta_i^T - B_i^{-1}\eta_i \mathbb{E}[L_i(1)]\mathbb{E}[L_i^T(1)]\eta_i^T B_i^{-T} \,. \tag{3.6}$$

This holds for every $i = 1, \ldots, m$.

From an empirical point of view, the covariance structures between factors in the temporal direction are useful. We compute this in the next Lemma:

Lemma 3.5. *Suppose that $L_i(1)$ is square integrable for $i = 1, \dots, m$. Then, for $0 \leq s \leq t$,*

$$\mathrm{Cov}[X(t), Y_i(t)|\mathcal{F}_s] = 0 = \mathrm{Cov}[Y_i(t), Y_j(t)|\mathcal{F}_s]\,,$$

for $i \neq j$ and $i, j = 1, \dots, m$. Furthermore, if $\widetilde{L}_j(1)$ are integrable for $j = 1, \dots, n$, then the conditional auto-covariance functions of X and Y_i are given by,

$$\begin{aligned}\mathrm{acov}_X(s,t,h) &:= \mathrm{Cov}[X(t+h), X(t)|\mathcal{F}_s] = \boldsymbol{e}^{Ah}\mathrm{Var}[X(t)|\mathcal{F}_s] \\ \mathrm{acov}_{Y_i}(s,t,h) &:= \mathrm{Cov}[Y_i(t+h), Y_i(t)|\mathcal{F}_s] = \boldsymbol{e}^{B_i h}\mathrm{Var}[Y_i(t)|\mathcal{F}_s]\,,\end{aligned}$$

for $i = 1, \dots, m$, $0 \leq s \leq t$ and $h \geq 0$ a constant (the lag of the auto-covariance).

Proof. First, from (3.2) we find,

$$\mathrm{Cov}[Y_i(t), Y_j(t)|\mathcal{F}_s] = \mathrm{Cov}\left[\int_s^t \mathbf{e}^{B_i(t-u)}\eta_i dL_i(u), \int_s^t \mathbf{e}^{B_j(t-u)}\eta_j dL_j(u)\right] = 0\,,$$

for $i \neq j$, since in that case L_i and L_j are independent.

Next, from (3.1) and (3.2) we find for given $i = 1, \dots, m$,

$$\mathrm{Cov}[X(t), Y_i(t)|\mathcal{F}_s] = \mathrm{Cov}\left[\int_s^t \mathbf{e}^{A(t-u)}\Sigma(u)^{1/2}dW(u), \int_s^t \mathbf{e}^{B_i(t-u)}\eta_i dL_i(u)|\mathcal{F}_s\right]\,.$$

We recall that $\Sigma(t)$ and $W(t)$ are independent. Using the tower property of conditional expectation, where we condition of the σ-algebra $\mathcal{G}_{t,s}$ generated by all paths of $\widetilde{L}_j(u); 0 \leq u \leq t$ and $\mathcal{F}_s$, for $j = 1, \dots, n$, we find,

$$\begin{aligned}&\mathbb{E}\left[\left(\int_s^t \mathbf{e}^{A(t-u)}\Sigma(u)^{1/2}dW(u)\right)\left(\int_s^t \mathbf{e}^{B_i(t-u)}\eta_i dL_i(u)\right)^T |\mathcal{F}_s\right] \\ &= \mathbb{E}\left[\mathbb{E}\left[\left(\int_s^t \mathbf{e}^{A(t-u)}\Sigma(u)^{1/2}dW(u)\right)\left(\int_s^t \mathbf{e}^{B_i(t-u)}\eta_i dL_i(u)\right)^T |\mathcal{G}_{t,s}\right]|\mathcal{F}_s\right]\,, \\ &= \mathbb{E}\left[\mathbb{E}\left[\left(\int_s^t \mathbf{e}^{A(t-u)}\Sigma(u)^{1/2}dW(u)\right)|\mathcal{G}_{t,s}\right]\left(\int_s^t \mathbf{e}^{B_i(t-u)}\eta_i dL_i(u)\right)^T |\mathcal{F}_s\right]\,, \\ &= 0\,.\end{aligned}$$

In the second equality we have used that $L_i(u)$ for $i = 1, \dots, m$ are the diagonals of $\widetilde{L}_i(u)$, and thus measurable with respect to $\mathcal{G}_{t,s}$, while the last equality follows since the expectation of an Itô integral is zero.

Next, let us derive the auto-covariance function for X. From (3.1), we find for $h \geq 0$

$$X(t+h) = \mathbf{e}^{Ah} X(t) + \int_t^{t+h} \mathbf{e}^{A(t+h-u)} \Sigma(u)^{1/2}\, dW(u)\,.$$

Hence,

$$\mathrm{acov}_X(s,t,h) = \mathbf{e}^{Ah}\mathrm{Var}[X(t)|\mathcal{F}_s] + \mathrm{Cov}[\int_t^{t+h} \mathbf{e}^{A(t+h-u)} \Sigma(u)^{1/2}\, dW(u), X(t)|\mathcal{F}_s]\,.$$

By using the same double conditioning argument as above, we see that the seocnd term is equal to zero since Brownian motion has indendependent increments. This proves the auto-covariance function of X. For the case Y_i, we use exactly the same argument. Use (3.2) and the independent increment property of Lévy processes to reach the result. The Lemma follows. □

From an empirical point of view, the stationary auto-covariance functions are particularly interesting. From (3.5) and (3.6) it follows

$$\lim_{t\to} \mathrm{acov}_X(s,t,h) = \mathbf{e}^{Ah} \sum_{j=1}^{n} \omega_j \mathbf{A}^{-1}\mathbf{C}_j^{-1}\mathbb{E}[\widetilde{L}_j(1)]\,,$$
$$\lim_{t\to} \mathrm{acov}_Y(s,t,h) = \mathbf{e}^{B_i h}\left\{\mathbf{B}_i^{-1}\eta_i \mathbb{E}[L_i(1)L_i^T(1)]\eta_i^T - B_i^{-1}\eta_i\mathbb{E}[L_i(1)]\mathbb{E}[L_i^T(1)]\eta_i^T\right\}\,.$$

As A and B_i have eigenvalues with negative real parts, we see that the de-seasonalized log-spot prices $\ln S_k(t) - \ln \Lambda_k(t)$ of commodity $k = 1,\dots,d$ will in stationarity have an auto-correlation function being a sum of exponential functions, with decay rates given by the real parts of the eigenvalues of A and B_i, $i = 1,\dots,m$. This is an empirical feature we often see with energy prices (see for example Benth, Kiesel and Nazarova [10]).

3.1 Cumulants and stationary distributions

Under the log integrability conditions (2.8), the processes Y_i and Z_j are stationary (see Sato [27], Thm. 5.2). In the next Proposition the characteristic function of the stationary distributions of X, Y_i and Z_j for $i = 1,\dots,m$ and $j = 1,\dots,n$ are calculated in terms of the characteristic function of the matrix-valued processes $\widetilde{L}_j$.

Let us first investigate the cumulant and the stationary distribution of Z_j and Y_i, $i = 1,\dots,m$, $j = 1,\dots,n$.

Proposition 3.6. *For $t \geq s$, the conditional cumulant functions of Y_i and Z_j are, resp.,*

$$\phi_{Y_i}^{(s,t)}(z) = \mathrm{i}\left(\boldsymbol{e}^{B_i(t-s)}Y_i(s) + \mathrm{i}B_i^{-1}(I - \boldsymbol{e}^{B_i(t-s)})\mu_i\right)^T z + \int_0^{t-s} \phi_{\widetilde{L}_i}(J_d(\eta_i^T \boldsymbol{e}^{B_i^T u} z))du\,, \quad (3.7)$$

$$\phi_{Z_j}^{(s,t)}(V) = \mathrm{i}V\boldsymbol{e}^{C_j(t-s)}Z_j(s)\boldsymbol{e}^{C_j^T(t-s)} + \int_0^{t-s} \phi_{\widetilde{L}_j}(\boldsymbol{e}^{C_j(t-s)}V\boldsymbol{e}^{C_j^T(t-s)})\, du\,, \tag{3.8}$$

for $i = 1, \ldots, m$ *and* $j = 1, \ldots, n$.

Proof. For the cumulants of Y_i, $i = 1, \ldots, m$ using (3.2) it holds

$$Y_i(t) = \mathbf{e}^{B_i(t-s)}Y^i(s) + B_i^{-1}(I - \mathbf{e}^{B_i(t-s)})\mu_i + \int_s^t \mathbf{e}^{B_i(t-u)}\eta_i dL_i(u)\,,$$

for $t \geq s$. Hence, by the key formula (see Sato [27]), the conditional cumulant function of $Y_i(t)$ given $\mathcal{F}_s$ is

$$\begin{aligned}\phi_{Y_i}^{(s,t)}(z) &= \mathrm{i}\left(\mathbf{e}^{B_i(t-s)}Y^i(s) + \mathrm{i}B_i^{-1}(I - \mathbf{e}^{B_i(t-s)})\mu^i\right)^T z + \int_s^t \phi_{L_i}(\eta_i^T \mathbf{e}^{B_i^T(t-u)}z)du\,,\\ &= \mathrm{i}\left(\mathbf{e}^{B_i(t-s)}Y_i(s) + \mathrm{i}B_i^{-1}(I - \mathbf{e}^{B_i(t-s)})\mu_i\right)^T z + \int_0^{t-s} \phi_{\widetilde{L}_i}(J_d(\eta_i^T \mathbf{e}^{B_i^T u}z))du\,.\end{aligned}$$

The cumulant functions of the Z_j's are computed in Pigorsch and Stelzer [23]. We include the derivation here for the convenience of the reader. By (3.3) and the independent increment property of Lévy processes,

$$\begin{aligned}\ln \mathbb{E}\left[\mathbf{e}^{\mathrm{i}VZ_j(t)}|\mathcal{F}_s\right] &= \mathrm{i}V\mathbf{e}^{C_j(t-s)}Z_j(s)\mathbf{e}^{C_j^T(t-s)} + \ln \mathbb{E}\left[\mathbf{e}^{\mathrm{i}V\int_s^t \mathbf{e}^{C_j(t-u)}\, d\widetilde{L}_j(u)\mathbf{e}^{C_j^T(t-u)}}|\mathcal{F}_s\right]\\ &= \mathrm{i}V\mathbf{e}^{C_j(t-s)}Z_j(s)\mathbf{e}^{C_j^T(t-s)} + \ln \mathbb{E}\left[\mathbf{e}^{\mathrm{i}V\int_s^t \mathbf{e}^{C_j(t-u)}\, d\widetilde{L}_j(u)\mathbf{e}^{C_j^T(t-u)}}\right]\\ &= \mathrm{i}V\mathbf{e}^{C_j(t-s)}Z_j(s)\mathbf{e}^{C_j^T(t-s)} + \int_s^t \phi_{\widetilde{L}_j}\left(\mathbf{e}^{C_j(t-u)}V\mathbf{e}^{C_j^T(t-u)}\right)\, du\,.\end{aligned}$$

Hence, the Lemma follows. □

Since $L(t)$ has finite log moments and $\sigma(B_i) \subseteq (-\infty, 0) + i\mathbb{R}^+$, the limit of $\phi_{Y_i}^{(s,t)}$ for $t \to \infty$ is well-defined (see Sato [27]) and given by

$$\lim_{t\to\infty} \phi^{(s,t)}\phi_{Y_i}(z) := \phi_{Y_i}(z) = i\mu_i^T(B_i^T)^{-1}z + \int_0^\infty \phi_{\widetilde{L}_i}(J_d(\eta_i^T e^{B_i^T u}z))du, \quad z \in \mathbb{R}^d\,,$$

for $i = 1, \ldots, m$. This is the cumulant function of the stationary distribution of Y_i. Similarily, we find the cumulant function of the stationary distribution of the Z_j's to be

$$\lim_{t\to\infty} \phi_{Z_j}^{(s,t)}(z) := \phi_{Z_j}(V) = \int_0^\infty \phi_{\widetilde{L}_j}\left(\mathbf{e}^{C_j^T s}V\mathbf{e}^{C_j s}\right) ds, \quad V \in \mathbb{S}_d\,,$$

for $j = 1, \ldots, n$.

Let us continue our analysis with deriving the cumulant function and characterize the stationary distribution of the base component X. To this end, we define the family of linear operators $\mathcal{C}_j(t)$,

$$\mathcal{C}_j(t) : X \mapsto \omega_j \left[(\mathbf{C_j} - \mathbf{A})^{-1} \left(e^{C_j t} X e^{C_j^T t} - e^{At} X e^{A^T t} \right) \right] . \tag{3.9}$$

We remark that a similar operator is defined in Pigorsch and Stelzer [23]. The following auxiliary result is useful:

Lemma 3.7. *Define* $f(s,t) := \int_s^t e^{A(t-u)} \Sigma(u) e^{A^T(t-u)} du$. *Then it holds*

$$f(s,t) = \sum_{j=1}^n \mathcal{C}_j(t-s) Z_j(s) + \int_s^t \mathcal{C}_j(t-v) d\widetilde{L}_j(v) ,$$

for $0 \le s \le t$.

Proof. Using (3.3) and the assumption that A and C_j commute for $j = 1, \ldots, n$ it holds

$$\begin{aligned}
f(s,t) &= \int_s^t \mathbf{e}^{A(t-u)} \sum_{j=1}^n \omega_j \left(\mathbf{e}^{C_j(u-s)} Z_j(s) \mathbf{e}^{C_j^T(u-s)} + \int_s^u \mathbf{e}^{C_j(u-v)} d\widetilde{L}_j(v) \mathbf{e}^{C_j^T(u-v)} \right) \mathbf{e}^{A^T(t-u)} du \\
&= \sum_{j=1}^n \omega_j \int_s^t \mathbf{e}^{(C_j - A)u} \mathbf{e}^{At - C_j s} \left(Z_j(s) + \int_s^u \mathbf{e}^{-C_j v} d\widetilde{L}_j(v) \mathbf{e}^{-C_j^T v} \right) \mathbf{e}^{A^T t - C_j^T s} \mathbf{e}^{(C_j - A)^T u} du \\
&= \sum_{j=1}^n \omega_j (\mathbf{C_j} - \mathbf{A})^{-1} \left(\mathbf{e}^{C_j(t-s)} Z_j(s) e^{C_j^T(t-s)} - \mathbf{e}^{A(t-s)} Z_j(s) \mathbf{e}^{A^T(t-s)} \right) \\
&\quad + \int_s^t \int_s^u \left\{ \mathbf{e}^{(C_j - A)u} \mathbf{e}^{At} \mathbf{e}^{-C_j v} d\widetilde{L}_j(v) \mathbf{e}^{-C_j^T v} \mathbf{e}^{A^T t} \mathbf{e}^{(C_j - A)^T u} \right\} du .
\end{aligned}$$

The last integral is intepreted as *first* integrating with respect to $d\widetilde{L}_j(v)$, and *next* integrating the obtained expression with respect to du. But, by spelling out the integrals in terms of sums, using the definition of the $d\widetilde{L}_j(v)$ integrals, and invoking the stochastic Fubini theorem (see Protter [24]), we get

$$\begin{aligned}
&\int_s^t \int_s^u \left\{ \mathbf{e}^{(C_j - A)u} \mathbf{e}^{At} \mathbf{e}^{-C_j v} d\widetilde{L}_j(v) \mathbf{e}^{-C_j^T v} \mathbf{e}^{A^T t} \mathbf{e}^{(C_j - A)^T u} \right\} du \\
&\qquad = \int_s^t \int_v^t \left\{ \mathbf{e}^{(C_j - A)u} \mathbf{e}^{At} \mathbf{e}^{-C_j v} d\widetilde{L}_j(v) \mathbf{e}^{-C_j^T v} \mathbf{e}^{A^T t} \mathbf{e}^{(C_j - A)^T u} \right\} du .
\end{aligned}$$

Here, the right hand side is intepreted as *first* integrating with respect to du, treating $d\widetilde{L}_j(v)$ as a matrix and not a differential, and *next* integrating with respect to $d\widetilde{L}_j(v)$ the obtained expression.

Hence, we find

$$f(s,t) = \sum_{j=1}^{n} \mathcal{C}_j(t-s)Z_j(s) \\ + (\mathbf{C_j} - \mathbf{A})^{-1} \left(\int_s^t \mathbf{e}^{C_j(t-v)} d\widetilde{L}_j(v) \mathbf{e}^{C_j^T(t-v)} - \int_s^t \mathbf{e}^{A(t-v)} d\widetilde{L}_j(v) \mathbf{e}^{A^T(t-v)} \right).$$

The Lemma follows. □

With this result at hand, we can derive the conditional cumulant function of $X(t)$. This is done in the next Proposition.

Proposition 3.8. *The conditional cumulant function of the process $X(t)$ given $\mathcal{F}_s$ is*

$$\phi_X^{s,t}(z) = \mathrm{i}X^T(s)\boldsymbol{e}^{A^T(t-s)}z - \frac{1}{2}\sum_{j=1}^{n} z^T \mathcal{C}_j(t-s)Z_j(s)z + \sum_{j=1}^{n}\int_0^{t-s} \phi_{\widetilde{L}_j}\left(\frac{1}{2}\mathrm{i}\mathcal{C}_j^*(u)zz^T\right) du\,, \tag{3.10}$$

for every $0 \le s \le t$, and $z \in \mathbb{R}^d$, where $\mathcal{C}_j^*$ is the adjoint operator of $\mathcal{C}_j$ defined in (3.9).

Proof. Let $\mathcal{G}_{t,s}$ denote the filtration generated by $\mathcal{F}_s$ and the paths $\Sigma(u), 0 \le u \le t$. By the independence of W and $\widetilde{L}_j$ for $j = 1 \ldots n$, and the tower property of conditional expectations, we have that

$$\begin{aligned} \phi_X^{s,t}(z) &= \ln \mathbb{E}\left[\mathbb{E}\left[\mathbf{e}^{\mathrm{i}\langle z, X(t)\rangle} | \mathcal{G}_{t,s}\right] | \mathcal{F}_s\right] \\ &= \mathrm{i}X^T(s)\mathbf{e}^{A^T(t-s)}z + \ln \mathbb{E}\left[\mathbb{E}\left[\exp\left(\mathrm{i}\left(\int_s^t \Sigma(u)^{1/2}\mathbf{e}^{A(t-u)}dW(u)\right)^T z\right) | \mathcal{G}_{t,s}\right] | \mathcal{F}_s\right] \\ &= \mathrm{i}X^T(s)\mathbf{e}^{A^T(t-s)}z + \ln \mathbb{E}\left[\exp\left(-\frac{1}{2}z^T\int_s^t \mathbf{e}^{A(t-u)}\Sigma(u)\mathbf{e}^{A^T(t-u)}du\; z\right) | \mathcal{F}_s\right] \end{aligned}$$

In the second equality, we used (3.1) and in the third equality we used the Gaussianity of a Wiener integral (note that the integrand is a deterministic function after conditioning on $\mathcal{G}_{t,s}$). Appealing to Lemma 3.7, we find

$$\phi_X^{s,t}(z) = \mathrm{i}X^T(s)\mathbf{e}^{A^T(t-s)}z - \frac{1}{2}\sum_{j=1}^{n} z^T\mathcal{C}_j(t-s)Z_j(s)z \\ + \sum_{j=1}^{n} \ln \mathbb{E}\left[\exp\left(-\frac{1}{2}z^T\int_s^t \mathcal{C}_j(t-u)d\widetilde{L}_j(u)z\right) | \mathcal{F}_s\right]$$

$$
\begin{aligned}
&= \mathrm{i}X^T(s)\mathbf{e}^{A^T(t-s)}z - \frac{1}{2}\sum_{j=1}^{n} z^T\mathcal{C}_j(t-s)Z_j(s)z \\
&\quad + \sum_{j=1}^{n} \ln \mathbb{E}\left[\exp\left(itr\left(\frac{1}{2}\mathrm{i}zz^T\int_s^t \mathcal{C}_j(t-u)d\widetilde{L}_j(u)\right)\right)\right] .
\end{aligned}
$$

In the last step, we used the fundamental relation $z^TUz = tr(zz^TU)$ for a quadratic matrix U together with the independent increment property of Lévy processes. Now, observe that the stochastic integral can be expressed as

$$
\int_s^t \mathcal{C}_j(t-u)d\widetilde{L}_j(u) = \lim_{|\Delta_i|\to 0}\sum_{i=0}^{n-1}\mathcal{C}_j(t-u_i)\Delta\widetilde{L}_j(u_i)\,,
$$

for partitions $s = u_0 < \cdots < u_n = t$ with $\Delta_i := \widetilde{L}_j(u_{i+1}) - \widetilde{L}_j(u_i)$ and $\Delta_i := u_{i+1} - u_i$. By independence of increments of a Lévy process, and continuity of the exponential function together with Fubini-Tonelli's Theorem, we get

$$
\begin{aligned}
&\mathbb{E}\left[\exp\left(\mathrm{i}tr\left(\frac{1}{2}\mathrm{i}zz^T\int_s^t \mathcal{C}_j(t-u)d\widetilde{L}_j(u)\right)\right)\right] \\
&\quad = \lim_{|\Delta_i|\to 0}\prod_{i=1}^{n-1}\mathbb{E}\left[\exp\left(itr\left(\frac{1}{2}\mathrm{i}zz^T\mathcal{C}_j(t-u_i)\Delta\widetilde{L}_j(u_i)\right)\right)\right] .
\end{aligned}
$$

Now, the linear operators $\mathcal{C}_j(t-u_i)$ can be represented as $\mathrm{vec}^{-1}\circ\mathcal{K}\circ\mathrm{vec}$ for a matrix $\mathcal{K}\in\mathbb{R}^{d^2}$. Hence, since for quadratic matrices $tr(VX) = \mathrm{vec}(V)^T\mathrm{vec}(X)$, we find

$$
\begin{aligned}
tr\left(V\mathcal{C}_j(t-u_i)\Delta\widetilde{L}_j(u_i)\right) &= \mathrm{vec}(V)^T\mathrm{vec}\left(\mathcal{C}_j(t-u_i)\Delta\widetilde{L}_j(u_i)\right) \\
&= \mathrm{vec}(V)^T\mathrm{vec}\left(\mathrm{vec}^{-1}\circ\mathcal{K}\circ\mathrm{vec}(\Delta\widetilde{L}_j(u_i))\right) \\
&= \mathrm{vec}(V)^T\mathcal{K}\mathrm{vec}(\Delta\widetilde{L}_j(u_i)) \\
&= \left(\mathcal{K}^T\mathrm{vec}(V)\right)^T\mathrm{vec}(\Delta\widetilde{L}_j(u_i))\,.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
\ln\mathbb{E}\left[\exp\left(itr(V\mathcal{C}_j(t-u_i)\Delta\widetilde{L}_j(u_i))\right)\right] &= \ln\mathbb{E}\left[\exp\left(\mathrm{i}\left(\mathcal{K}^T\mathrm{vec}(V)\right)^T\mathrm{vec}(\Delta\widetilde{L}_j(u_i))\right)\right] \\
&= \ln\mathbb{E}\left[\exp\left(itr\left(\mathrm{vec}^{-1}(\mathcal{K}^T\mathrm{vec}(V))\Delta\widetilde{L}_j(u_i)\right)\right)\right] \\
&= \phi_{\widetilde{L}_j}\left(\mathrm{vec}^{-1}\circ\mathcal{K}^T\circ\mathrm{vec}(V)\right)\Delta_i \\
&= \phi_{\widetilde{L}_j}\left(\mathcal{C}_j^*(t-u_i)V\right)\Delta_i\,.
\end{aligned}
$$

Letting $V = \frac{1}{2}\mathrm{i}zz^T$, we conclude that

$$
\begin{aligned}
\ln \mathbb{E}\left[\exp\left(\mathrm{i}tr\left(\frac{1}{2}\mathrm{i}zz^T\mathcal{C}_j(t-u_i)\Delta\widetilde{L}_j(u_i)\right)\right)\right] &= \int_s^t \phi_{\widetilde{L}_j}\left(\frac{1}{2}\mathrm{i}\mathcal{C}_j^*(t-u)zz^T\right)du \\
&= \int_0^{t-s} \phi_{\widetilde{L}_j}\left(\frac{1}{2}\mathrm{i}\mathcal{C}_j^*(u)zz^T\right)du\,.
\end{aligned}
$$

This proves the Proposition. $\square$

We can prove the stationarity of $X(t)$ and derive the cumulant function for the limiting distribution.

Proposition 3.9. *The process $X(t)$ is stationary and the cumulant function of the limiting distribution is given by*

$$
\lim_{t\to\infty} \phi_X^{(s,t)}(z) := \phi_X(z) = \sum_{j=1}^n \int_0^\infty \phi_{\widetilde{L}_j}\left(\frac{1}{2}\mathrm{i}\mathcal{C}_j^*(s)zz^T\right)ds\,,
$$

where $z \in \mathbb{R}^d$ and the linear operator $\mathcal{C}_j(t)$ is defined in (3.9).

Proof. By the definition of $\mathcal{C}_j(t)$ and the fact that A and C_j, $j = 1,\ldots,n$ have eigenvalues with negative real parts, it is straightforward to see that

$$
\lim_{t\to\infty} \mathrm{i}X^T(s)\mathbf{e}^{A^T(t-s)}z - \frac{1}{2}\sum_{j=1}^n z^T\mathcal{C}_j(t-s)Z_j(s)z = 0\,.
$$

Hence, we must prove that the integral

$$
\int_0^t \phi_{\widetilde{L}_j}\left(\frac{1}{2}\mathrm{i}\mathcal{C}_j^*(s)zz^T\right)ds
$$

converges when $t \to \infty$, for every $j = 1,\ldots,n$. To prove this it is sufficient to show that

$$
\int_s^t \mathcal{C}_j(t-u)d\widetilde{L}_j(u)
$$

has a stationary solution for each $j = 1,\ldots,n$. Let us fix $j = 1,\ldots,n$, and observe that by definition of $\mathcal{C}_j(t)$ and linearity of the $\mathbf{C}_j - \mathbf{A}$-operator, we have

$$
\int_s^t \mathcal{C}_j(t-u)d\widetilde{L}_j(u) = \omega_j(\mathbf{C}_j-\mathbf{A})^{-1}\left\{\int_s^t \mathbf{e}^{C_j(t-u)}\,d\widetilde{L}_j(u)\mathbf{e}^{C_j(t-u)} - \int_s^t \mathbf{e}^{A(t-u)}\,d\widetilde{L}_j(u)\mathbf{e}^{A(t-u)}\right\}\,.
$$

But the two stochastic integrals are stationary by Sato [27] Theorem 5.2 since A and C_j have eigenvalues with negative real parts. Hence, the result follows since any linear combination of stationary processes will in itself be stationary. □

We observe that the limiting distribution of X must be centered and symmetric since its cumulant function satisfies $\phi_X(z) = \phi_X(-z)$. We discuss the stationary distribution of X in more detail.

As we now argue, the stationary distribution of X can be viewed as the convolution of a centered normal and a leptokurtic distribution whenever $\widetilde{L}_j(1)$ are integrable for $j = 1, \ldots, n$. To show this we introduce the zero-mean matrix valued Lévy process $\widehat{L}_j(t) \triangleq \widetilde{L}_k(t) - \mathbb{E}[\widetilde{L}_j(1)]t$, and denote by $\phi_{\widehat{L}_j}(V)$ its cumulant defined by

$$\phi_{\widehat{L}_j}(V) = \phi_{\widetilde{L}_j}(V) - \mathrm{i}tr(V\mathbb{E}[\widetilde{L}_j(1)])\,.$$

The cumulant function of the stationary distribution of $X(t)$ can henceforth be expressed as

$$\begin{aligned}\phi_X(z) &= \sum_{j=1}^{n}\left\{\int_0^\infty \phi_{\widehat{L}_j}\left(\frac{1}{2}\mathrm{i}\mathcal{C}_j^*(s)zz^T\right)ds + \mathrm{i}\int_0^\infty tr\left(\frac{1}{2}\mathrm{i}\mathcal{C}_j^*(s)zz^T\,\mathbb{E}[\widetilde{L}_j(1)]\right)ds\right\}\\ &= \sum_{j=1}^{n}\left\{\int_0^\infty \phi_{\widehat{L}_j}\left(\frac{1}{2}\mathrm{i}\mathcal{C}_j^*(s)zz^T\right)ds - \frac{1}{2}\int_0^\infty tr\left(\mathcal{C}_j^*(s)zz^T\mathbb{E}[\widetilde{L}_j(1)]\right)ds\right\}.\end{aligned}$$

Using properties of the trace-operator we have

$$\begin{aligned}tr\left((\mathcal{C}_j^*(s)zz^T)\mathbb{E}[\widetilde{L}_j(1)]\right) &= \mathrm{vec}(\mathcal{C}_j^*(s)zz^T)^T\mathrm{vec}(\mathbb{E}[\widetilde{L}_j(1)])\\ &= \mathrm{vec}\left(\mathrm{vec}^{-1}(\mathcal{K}^T\mathrm{vec}(zz^T))\right)^T\mathrm{vec}(\mathbb{E}[\widetilde{L}_j(1)])\\ &= \left(\mathcal{K}^T\mathrm{vec}(zz^T)\right)^T\mathrm{vec}(\mathbb{E}[\widetilde{L}_j(1)])\\ &= \mathrm{vec}(zz^T)\mathcal{K}\mathrm{vec}(\mathbb{E}[\widetilde{L}_j(1)])\\ &= tr\left(zz^T\mathrm{vec}^{-1}(\mathcal{K}\mathrm{vec}(\mathbb{E}[\widetilde{L}_j(1)]))\right)\\ &= tr(zz^T\mathcal{C}_j(s)\mathbb{E}[\widetilde{L}_j(1)])\\ &= z^T\mathcal{C}_j(s)\mathbb{E}[\widetilde{L}_j(1)]z\,.\end{aligned}$$

Here, we have used that the operator $\mathcal{C}_j(s)$ can be represented by the $\mathbb{R}^{d^2\times d^2}$-matrix $\mathcal{K}$ as $\mathcal{C}_j(s) = \mathrm{vec}^{-1}\circ\mathcal{K}\circ\mathrm{vec}$. Using Lemma 3.3, we conclude

$$\phi_X(z) = \sum_{j=1}^{n}\int_0^\infty \phi_{\widehat{L}_j}\left(\frac{1}{2}\mathrm{i}\mathcal{C}_j^*(s)zz^T\right)ds - \frac{1}{2}z^T\left(\lim_{t\to\infty}\mathrm{Var}[X(t)]\right)z\,.$$

The last term is the characteristic function of a centered multivariate normal distribution with variance equal to $\lim_{t\to\infty} \mathrm{Var}[X(t)]$. We remark that this coincides with the stationary distribution obtained from the multivariate Schwartz model having constant volatility $\Sigma \in M_d(\mathbb{R})$ given by

$$\Sigma \triangleq \lim_{t\to\infty} \mathrm{Var}[X(t)]\,.$$

The first term in $\phi_X(z)$ will be the characteristic function of a non-Gaussian distribution.

4 Analysis of the spot dynamics

Let us look at the dynamics of $\widetilde{S}(t) \triangleq S(t)/\Lambda(t)$, the deseasonalized spot price, where the division is done elementwise.

Proposition 4.1. *It holds that*

$$d\ln\widetilde{S}(t) = \Big(M(t) + A\ln\widetilde{S}(t)\Big)\,dt + \Sigma(t)^{1/2}\,dW(t) + \sum_{i=1}^{m}\eta_i\,dL_i(t)\,,$$

where

$$M(t) = \sum_{i=1}^{m}\mu_i + (-A + B_j)Y_j(t)\,.$$

Proof. This follows from rewriting the equations in (2.2) and (2.3). □

We see from this result that the dynamics can be interpreted as a mean-reverting process towards a stochastic mean. The mean will be described by the multivariate process $M(t)$, which will consist of linear combinations of the different "spike" components Y_j. The matrix A describes the "speed" of mean-reversion, as well as how the different commodities are functionally dependent on each other. Moreover, the stochastic volatility term and the spike contributions are clearly dependent.

We move on analysing the stationary distribution of $\ln\widetilde{S}(t)$. From Lemma 3.2, we find in stationarity that

$$\lim_{t\to\infty}\mathbb{E}[\ln\widetilde{S}(t)] = \lim_{t\to\infty}\mathbb{E}[X(t)] + \sum_{i=1}^{m}\mathbb{E}[Y_i(t)] = \sum_{i=1}^{m}B_i^{-1}\left(\mu_i + \eta_i\mathbb{E}[L_i(1)]\right)\,.$$

Furthermore, from Lemma 3.5 we know that in stationarity, the auto-covariance function of $\ln\widetilde{S}(t)$ is

$$acov_{\ln\widetilde{S}}(h) = acov_X(h) + acov_{\sum Y_i}(h) \tag{4.1}$$

$$
\begin{aligned}
&= \mathbf{e}^{A|h|} \sum_{j=1}^{n} \omega_j \mathbf{A}^{-1} \mathbf{C_j}^{-1} \mathbb{E}[\widetilde{L}_j(1)] \\
&\quad + \sum_{i=1}^{m} \mathbf{e}^{B_i|h|} \left(\mathbf{B_i}^{-1} \eta_i \mathbb{E}[L_i(1) L_i^T(1)] (\eta_i)^T - (B_i^{-1} \eta_i \mathbb{E}[L_i(1)] \mathbb{E}[L_i^T(1)] \eta_i^T \right) .
\end{aligned}
$$

Hence, in stationarity, the auto-covariance function of $\ln \widetilde{S}(t)$ will be a linear combination of exponentially decaying functions due to eigenvalues with a negative real part. This is in line with empirical observations of power prices, as we have earlier noted (see *e.g.* Benth, Kiesel and Nazarova [10]).

By combining the results of the cumulant functions for the different factors in the dynamics of $\ln \widetilde{S}(t)$ derived in the previous section, we can compute the cumulant of the deseasonalized log-spot prices. This is presented in the next Proposition.

Proposition 4.2. *The characteristic function of the stationary distribution of* $\ln \widetilde{S}(t)$ *is given by*

$$
\begin{aligned}
\phi_{\ln \widetilde{S}}(z) &= \sum_{i=1}^{m} \mathrm{i} \mu_i^T (B_i^T)^{-1} z + \sum_{j=1}^{n} \int_0^\infty \phi_{\widetilde{L}_j} \left(\frac{1}{2} \mathrm{i} \mathcal{C}_j^*(u) z z^T \right) du \\
&\quad + \sum_{i=1}^{m} \int_0^\infty \phi_{\widetilde{L}_i} \left(\frac{1}{2} \mathrm{i} \mathcal{C}_i^*(u) z z^T + J_d(\eta_i^T \boldsymbol{e}^{B_i^T u} z) \right) - \phi_{\widetilde{L}_i} \left(\frac{1}{2} \mathrm{i} \mathcal{C}_i^*(u) z z^T \right) du ,
\end{aligned}
$$

for $z \in \mathbb{R}^d$.

Proof. By combining Proposition 3.8 and equation (3.7) in Proposition 3.6 the conditional cumulant function of $\ln \widetilde{S}$ given $\mathcal{F}_s$ is

$$
\begin{aligned}
\phi_{\ln \widetilde{S}}^{s,t}(z) &= \mathrm{i} X^T(s) e^{A^T(t-s)} z - \frac{1}{2} \sum_{j=1}^{n} z^T \mathcal{C}_j(t-s) Z_j(s) z \\
&\quad + \sum_{i=1}^{m} \mathrm{i} Y^T(s) e^{B_i^T(t-s)} z + \mathrm{i} (B_i^{-1} (I - e^{B_i(t-s)}) \mu_i)^T z \\
&\quad + \sum_{i=1}^{m} \int_0^{t-s} \phi_{\widetilde{L}_i} \left(\frac{1}{2} \mathrm{i} \mathcal{C}_i^*(u) z z^T + J_d(\eta_i^T e^{B_i^T u} z) \right) du \\
&\quad + \sum_{j=m+1}^{n} \int_0^{t-s} \phi_{\widetilde{L}_j} \left(\frac{1}{2} \mathrm{i} \mathcal{C}_j^*(u) z z^T \right) .
\end{aligned}
$$

Since a stationary solution exists for X and all Y_i's, there also exists a stationary solution for $\ln \widetilde{S}$. The Proposition follows by taking limits for $t \to \infty$ using that the real parts of the eigenvalues of the involved matrices are negative. □

Note that the sum over j in the expression for $\phi_{\ln \widetilde{S}}$ is stemming from the stationary cumulant of X, and therefore is from a symmetric centered random variable. Stationarity is a desirable feature in commodity markets being a reflection of supply and demand-driven prices. However, many studies argue for non-stationary effects (like for example Burger *et al.* [13] studying German electricity spot prices). We can easily extend our model to include non-stationary factors, like for instance choosing one or more of the Y's to be drifted Brownian motions rather than Ornstein-Uhlenbeck processes. We shall not discuss these modelling issues further here, but leave the analysis of this to the interested reader.

In the special case of a multivariate stochastic volatility Schwartz model (i.e. $m = 0$) the "reversion-adjusted" logreturns are approximately distributed according to a multivariate mean-variance mixture model. Considering the "reversion-adjusted" logreturns over the time interval $[t, t + \tau]$, we find

$$\begin{aligned} \ln \widetilde{S}(t+\tau) - \mathbf{e}^{A\tau} \ln \widetilde{S}(t) &= X(t+\tau) - \mathbf{e}^{A\tau} X(t) \\ &= \int_t^{t+\tau} \mathbf{e}^{A(t+\tau-s)} \Sigma^{1/2}(s)\, dW(s) \\ &\approx \mathbf{e}^{A\tau} \Sigma^{1/2}(t) \Delta_\tau W(t)\,. \end{aligned}$$

Here, $\Delta_\tau W(t) \triangleq W(t+\tau) - W(t)$ and τ is supposed to be sufficiently small in order to make the approximation above reasonable. Hence, we have that "reversion-adjusted" logreturns are approximately distributed according to the multivariate mean-variance mixture model

$$\mathbf{e}^{A\tau} \Sigma^{1/2}(t) \Delta_\tau W(t) \Big| |_{\Sigma(t)} \sim \mathcal{N}(0, \mathbf{e}^{A\tau} \Sigma(t) \mathbf{e}^{A^T \tau})\,.$$

In Benth [7], this was discussed in the univariate case, showing that we can choose stochastic volatility models yielding for instance normal inverse Gaussian distributed "reversion-adjusted" returns. We refer to Benth and Saltyte-Benth [8] for a study of gas and oil prices where the normal inverse Gaussian distribution has been applied to model "reversion-adjusted" returns. We further note that the conditional Gaussian structure of the "reversion-adjusted" returns implies that the covariance is determining the cross-commoditity dependency. In this case it is given explicitly by the stochastic volatility model $\Sigma(t)$, introducing a time-dependency in the covariance between commodities. In addition, the common factors $Y_i(t)$, $i = 1, \ldots, m$ will give co-dependent paths determined by common jump paths. Hence, we can mix rather complex dependency into the modelling. The auto-covariance function of the de-seasonalized logarithmic spot (4.1) gives explicit formulation to this dependence in terms of second order structure. For $h = 0$ the auto-covariance of de-seasonalized logarithmic spots gives the covariance matrix of the de-seasonalized logarithmic spots.

Let us discuss possible specifications of our spot price model satisfying the fundamental conditions on the operators and matrices in question. First of all, it is easily seen that if either one or both of the matrices A and C_j are diagonal, then they will commute. In fact, supposing that A is a diagonal matrix could be natural in view of interpreting the speed of mean reversion of each commodity modelled separately (as the corresponding entry on the diagonal), and not imposing any functional cross dependencies between the commodities. In such a model, dependencies will enter via the spike terms and in the stochastic volatility. If A is diagonal, then all the diagonal elements must be negative in order to have negative eigenvalues (eigenvalues are equal to the diagonal elements, of course). It is simple to see that the determinant of $A \otimes I + I \otimes A$ becomes

$$\det(A \otimes I + I \otimes A) = 2^d \prod_{i=1}^{d} a_i \prod_{i \neq j}^{d} (a_i + a_j)^2 \,,$$

which is unequal to zero since all the diagonal elements are strictly less than zero. This means that $\mathbf{A}$ is invertible. In fact, if we suppose that $\mathbf{C}_j$ also is diagonal, one finds that $\mathbf{A} - \mathbf{C}_j$ is invertible if and only if $a_i + a_j \neq c_i + c_j$ for $i, j = 1, \ldots, d$. Note also that stationarity of the volatility holds only if all the diagonals of C_j are strictly negative. But this also implies that $\mathbf{C}_j$ is invertible.

5 Simulation of matrix-valued subordinators

In this section we discuss simulation of our spot price dynamics, which essentially means to discuss simulation of matrix-valued subordinators.

Limited literature is available on the simulation of matrix-valued subordinators. One possible approach could be to apply existing methods to sample multivariate Lévy processes based on their Lévy measures by iterative sampling from the conditional marginals (see *e.g.* Cont and Tankov [14]). However, the marginal distribution functions are required, which are not always available in a simple form. Moreover, in case of matrix-valued subordinators, the restriction of the domain to the positive definite cone makes it even more complicated. We introduce a simple approximative algorithm[2] to simulate from matrix-valued compound Poisson, stable, and tempered stable processes with stable or constant jump-size distribution.

For any $U \in \mathbb{S}_d^+$ one can make a polar decomposition in a ray $r = ||U|| = tr(U^T U)^{1/2}$ and angle $\Theta = U/r$, so that $U = r\Theta$. Moreover, Θ is situated on the unit sphere $\mathbf{S}$ of $\mathbb{R}^{d\times d}$ intersected with the positive definite cone, *i.e.* $\Theta \in \mathbf{S}\mathbb{S}_d^+ \triangleq \mathrm{vec}^{-1}\mathbf{S} \cap \mathbb{S}_d^+$.

Suppose that ν is a Lévy measure on $\mathbb{S}_d^+$ of the subordinator L, such that it can be decomposed into

$$\nu(dU) = \Gamma(d\Theta)\widetilde{\nu}(\Theta, dr)\,, \quad U \in \mathbb{S}_d^+\,,$$

where $\widetilde{\nu}(\Theta, dr)$ is a Lévy measure on $\mathbb{R}_+$ and Γ is a spectral measure on $\mathbf{S}\mathbb{S}_d^+$ concentrated on a finite number of points $\{\Theta_i\}_{1\leq i\leq p}$. Note in passing that any measure can be approximated by a

[2]The idea of the algorithm was kindly proposed to us by Robert Stelzer.

measure concentrated on finitely many points. Since L is a pure-jump subordinator, its cumulant function is given by

$$\begin{aligned}\phi_{L(1)}(V) &= \int_{\mathbb{S}_d^+\setminus\{0\}} \left(\mathbf{e}^{\mathrm{i}\,tr(VU)}-1\right)\nu(dU)\\ &= \int_{\mathbf{S}\mathbb{S}_d^+}\int_0^\infty \left(\mathbf{e}^{\mathrm{i}rtr(V\Theta)}-1\right)\,\widetilde{\nu}(\Theta,dr)\,\Gamma(d\Theta)\\ &= \sum_{i=1}^p \Gamma(\Theta_i)\int_0^\infty \left(\mathbf{e}^{\mathrm{i}rtr(V\Theta_i)}-1\right)\,\widetilde{\nu}(\Theta_i,dr)\,.\end{aligned}$$

One recognizes this as the cumulant of a weighted sum of p independent real-valued subordinator processes. This leads to the following simple algorithm to sample L according to its cumulant function:

- Find the finite set of points $\{\Theta\}_{1\leq i\leq n}$ where Γ is concentrated.
- Simulate p independent subordinators $R_i(t)$ with cumulant function
$$\phi_{R_i(1)}(tr(V\Theta_i)) = \int_0^\infty \left(\mathbf{e}^{\mathrm{irtr}(V\Theta_i)}-1\right)\,\widetilde{\nu}(\Theta_i,dr)\,.$$
- Set $L(t)=\sum_{i=1}^p R_i(t)\Theta_i$.

To make this algorithm operationable, we must be able to sample the R_i's, which we now discuss in particular cases which are of interest in energy markets.

First, let us consider a matrix-valued compound process (mCP) with only positive jumps L. This becomes a multivariate compound Poisson process restricted to values in the symmetric positive definite cone. Its cumulant function is

$$\phi_{L(1)}(V) = \lambda\int_{\mathbb{S}_d^+}\left(\mathbf{e}^{\mathrm{itr}(VU)}-1\right)\,\nu(dU)\,,$$

where ν is the jump size distribution and λ the intensity. Supposing that $\nu(dU)=\widetilde{\nu}(\Theta,dr)\Gamma(\Theta)$ with $\widetilde{\nu}(\Theta,dr)$ being a probability distribution on $\mathbb{R}_+$ and $\Gamma(d\Theta)$ for a spectral measure Γ on $\mathbf{S}\mathbb{S}_d^+$, concentrated on finitely many points, it holds

$$\phi_{L(1)}(V)t = \lambda\sum_{i=1}^p\Gamma(\Theta_i)\int_0^\infty\left(\mathbf{e}^{\mathrm{i}rtr(V\Theta_i)}-1\right)\widetilde{\nu}(\Theta_i,dr)\,.$$

Hence, R_i for $i=1,\ldots,p$ will follow a one-dimensional compound Poisson process with jump intensity $\lambda\Gamma(\Theta_i)$ and jump-size distribution $\widetilde{\nu}(\Theta_i,dr)$. The $mCP(\lambda)$ process L is represented as a linear combination of angles Θ_i and radius processes being one-dimensional compound Poisson processes R_i, i.e. $L(t) = \sum_{i=1}^{p} R_i(t)\Theta_i$.

By exponential tilting of matrix-valued α-stable laws a multivariate extension of tempered stable laws can be made. The inverse Gaussian distribution is a special case of this class of functions. The polar decomposition of the Lévy measure ν of a matrix-valued tempered $\alpha/2$-stable law is given by (Barndorff-Nielsen and Pérez-Abreu [2])

$$\nu(dU) = \frac{\mathbf{e}^{-rtr(\Delta\Theta)}}{r^{1+\alpha/2}} dr\, \Gamma(d\Theta)\,.$$

In case $\alpha = 1$ then ν is a Lévy measure of a matrix extension of the inverse Gaussian distribution (mIG), where $\Delta \in \mathbb{S}_d^+$ and Γ, a finite measure on $\mathbf{S}\mathbb{S}_d^+$, are parameters. As in the univariate case the inverse Gaussian process is a pure jump process, hence the cumulant function is given by

$$\phi_{L(1)}(V) = \int_{\mathbf{S}\mathbb{S}_d^+} \int_0^\infty \left(\mathbf{e}^{\mathrm{i}r\,tr(V\Theta)} - 1\right)\mathbf{e}^{-rtr(\Delta\Theta)} \frac{\mathrm{d}r}{r^{3/2}}\, \Gamma(\mathrm{d}\Theta) + \mathrm{i}tr(V\mu_0)\,,$$

for $L(1) \sim mIG(\Delta, \Gamma, \mu_0)$, where $\mu_0 \in \mathbb{S}_d^+$ is a parameter. Choosing Γ such that it is concentrated on finitely many point and decomposing μ_0 in an angle $\Theta_0 \in \mathbf{S}\mathbb{S}_d^+$ and a radius $r_0 \in \mathbb{R}$ leads to

$$\phi_{L(1)}(V) = \sum_{i=1}^{p} \Gamma(\Theta_i) \int_0^\infty \left(\mathbf{e}^{\mathrm{i}rtr(V\Theta_i)} - 1\right)\mathbf{e}^{-rtr(\Delta\Theta_i)} \frac{\mathrm{d}r}{r^{3/2}} + \mathrm{i}r_0 tr(\Phi\Theta_0)\,.$$

One can compare this with the characteristic function of an one-dimensional inverse Gaussian random variable G, for which the cumulant function is given by

$$\phi_G(\zeta) = \mathrm{i}\frac{\delta}{\gamma}(2\mathcal{N}(\gamma) - 1)\zeta + \frac{\delta}{\sqrt{2\pi}} \int_0^\infty (\mathbf{e}^{\mathrm{i}\zeta x} - 1)\mathbf{e}^{-1/2\gamma^2 x} \frac{dx}{x^{3/2}} \quad \zeta \in \mathbb{R}\,.$$

where $\mathcal{N}$ denotes the cumulative normal distribution. We recognize L as a matrix of linear combinations of a finite number of angles Θ_i, $i = 1, \ldots, p$ with coefficients given by one-dimensional inverse Gaussian subordinator processes $R_i(t)$, where $R_i(1)$ is distributed according to the inverse Gaussian distribution $IG(\delta_i, \gamma_i)$, where $\delta_i = \sqrt{2\pi}\,\Gamma(\Theta_i)$ and $\gamma_i = 2\sqrt{tr(\Delta\Theta_i)}$. Moreover the drift parameter μ_0 of the multivariate inverse Gaussian distribution is by default chosen such that the drift term of the mIG distribution equals the drift term of $\sum_{i=1}^{p} R_i(t)\Theta_i$.

As an example, consider the case of two spot prices $S_1(t)$ and $S_2(t)$ modelled by our dynamics. For example, we could think of the spot price of electricity in two interconnected markets, or the spot price of gas and electricity. We suppose that the prices are driven by *two* $M_2(\mathbb{R})$-valued subordinator processes $\widetilde{L}_1(t), \widetilde{L}_2(t)$. The first process defines the spike component, while the second part is determining the stochastic volatility. We assume that there is *one* spike component $Y(t) \in \mathbb{R}^2$, while the stochastic volatility process $\Sigma(t)$ is the equally weighted sum of two processes $Z_1(t)$ and $Z_2(t)$, where the dynamics is driven by $\widetilde{L}_1$ and $\widetilde{L}_2$, resp. The dynamics of the spike process $Y(t)$ is driven by the diagonal of $\widetilde{L}_1(t)$. In order to make simulations from the model, we use specifications of the parameters in the model inspired by Vos [31], where the BNS stochastic volatility model was estimated to stock price data observed on the Dutch stock exchange. For simplification, we set the seasonality function equal to one, that is, $\Lambda_i(t) = 1$ for $i = 1, 2$. Moreover, we choose

$$A = \begin{pmatrix} -1.4 & -0.3 \\ -0.3 & -1.4 \end{pmatrix} \qquad B = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix} \qquad \eta = \begin{pmatrix} 1 & 0.5 \\ 0.5 & 1 \end{pmatrix}$$

$$C_1 = \begin{pmatrix} -0.4 & 0.3 \\ 0.3 & -0.4 \end{pmatrix} \qquad C_2 = \begin{pmatrix} -0.045 & 0.03 \\ 0.03 & -0.045 \end{pmatrix}$$

We let the levels of the spike component $Y(t)$ be zero, $\mu_1 = \mu_2 = 0$.

Next, let us define the two subordinator processes $\widetilde{L}_1(t)$ and $\widetilde{L}_2(t)$. To mimic spikes in the market, we consider a simple Poisson process for $\widetilde{L}_1(t)$. To have a stochastic volatility process which can generate adjusted returns being close to NIG distributed, we suppose that $\widetilde{L}_2(t)$ is mIG. In order to be able to simulate these two processes, we apply the idea above, and define a simple discrete spectral measure on $\mathbf{S}\mathbb{S}_2^+$. It is simple to see that

$$\Theta = \begin{pmatrix} \theta & \pm\sqrt{\theta(1-\theta)} \\ \pm\sqrt{\theta(1-\theta)} & 1-\theta \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} \theta & 0 \\ 0 & \sqrt{1-\theta^2} \end{pmatrix},$$

for $\theta \in (0, 1)$. There are three valid choices of $\Theta \in \mathbf{S}\mathbb{S}_2^+$. To this end, we discretize the unit interval with step size 0.1, and choose $\theta_j = j \times 0.1$ for $j = 1, \ldots, 9$. We choose either one of the three possible matrix structures for Θ with given θ_i, making up a total of 27 matrices Θ_i. For the Poisson process, we choose the intensity such that $\lambda\Gamma(\Theta_i) = 3/100$ and the jump size distribution set fixed to be 1.7, that is, if $R_i(t)$ is jumping at time t, then $\Delta R_i(t) = 1.7$. This will correspond to a change in spot price of a factor $\exp(1.7) = 5.47$, which is a rather dramatic price change. As a measure for the mIG part, we set $\Gamma(\Theta_i) = 1/324\sqrt{2\pi}$ uniformly for all $1 \leq i \leq 27$. Finally, we suppose that the parameter Δ of the mIG part is

$$\Delta = \begin{pmatrix} 50 & 45 \\ 45 & 50 \end{pmatrix}.$$

In Figure 1 the spot price series resulting from our 2-dimensional example is shown, where we have used an Euler scheme to discretize the dynamics in time and standard schemes for the sampling of inverse Gaussian distributions (see Rydberg [25]). One clearly can see the dependency between the two spot prices, in particular, how the spikes follow each other in the two series.

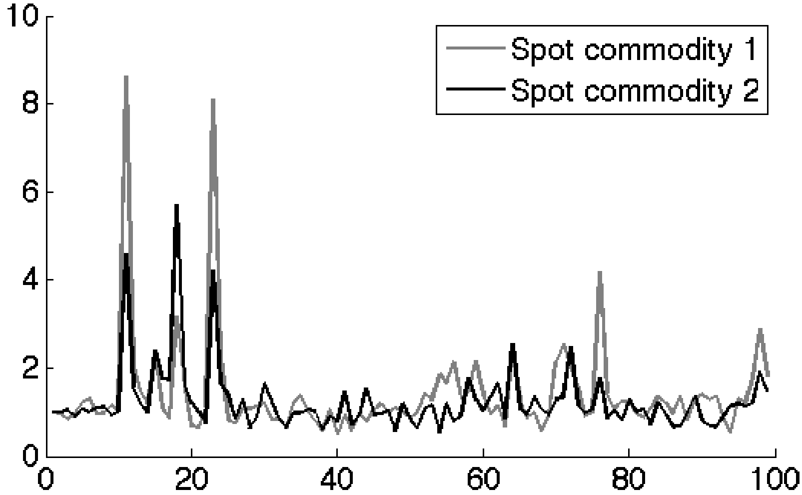

Figure 3.1: Simulated spot prices of the two commodities

6 Conclusions

We have proposed a model to describe the spot price dynamics for cross-commodity markets in a multivariate setting. The model captures features like mean-reversion, spikes, stochastic volatility, and inverse leverage effect. The dynamic is a multi-dimensional extension of the Barndorff-Nielsen and Shephard stochastic volatility model embedded into mean-reversion dynamics. This is relevant for commodity price series. The choice of the multi-dimensional extension is influenced by the work of Stelzer [29]. The multivariate spot model is analytically tractable and probabilistic properties can to a large extent be explicitly computed. We have derived various characteristics like stationary distributions and covariance functions. The model is a multivariate extension of the one-dimensional spot price dynamics analysed in Benth [7].

A simple algorithm to simulate from matrix-valued subordinators is introduced. The method is demonstrated on an emprical example. However, further research has to be done to generate matrix-valued Lévy processes in a more general setting, a study we leave for the future.

No methods exists to estimate the model based on spot price data. It is obviously of crucial interest for the applicability of the model to understand how to fit the parameters to data. Methods are available to estimate the model in the diffusion case on the quadratic covariation [5]. However these methods require high frequency data, which does not exist in the energy market. Another alternative is to adopt the methods already available for filtering spike data from price series into a multidimensional setting. If this is possible then the estimation of the spike process can be treated seperately from the diffusion part, and the diffusion part can be estimated conditionally on the spike parameters. Before this can be implemented further research has to be done on the validity of these methods. Another possibility is to estimate the parameters directly using the characteristic function in the frequency domain.

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Chapter 4

Pricing of forwards and options in a multivariate non-Gaussian stochastic volatility model for energy markets

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Abstract

In Benth and Vos [7] we introduced a multivariate spot price model with stochastic volatility for energy markets which captures characteristic features like price spikes, mean-reversion, stochastic volatility and inverse leverage effect as well as dependencies between commodities. In this paper we derive the forward price dynamics based on our multivariate spot price model, providing a very flexible structure for the forward curves, including contango, backwardation and hump shape. Moreover, a Fourier transform-based method to price options on the forward is described.

1 Introduction

The last decades the energy markets have been liberalized world-wide, resulting in market-places for commodities such as electricity, gas and coal. There are several markets for each of these commodities, geographically spread over the continents. For example in Europe we have markets for power in the UK, Germany, France, and the Nordic countries, to mention a few. There are transmission lines which interconnect these markets for electricity. Furthermore, since coal and gas are used to a large extent as fuels for power production, the prices for these commodities naturally affect the power prices. These markets become more and more integrated, both within one commodity, but also across the commodities. For this reason there is an increasing interest in studying multivariate models for energy markets, including cross-commodity models (like for example for gas, coal and electricity-markets) or multivariate models for the same commodity traded in different, but integrated markets (like for example the power markets in the Nordic countries and Germany).

In Benth and Vos [7] we propose stochastic dynamics for cross-commodity spot price modelling generalizing the univariate dynamics studied in Benth [5]. The model is flexible enough to capture spikes and mean-reversion. Moreover, it includes the possibility to model inverse leverage and stochastic volatility. The proposed dynamics can model co- and independent jump behaviour (spikes) in cross-commodity markets, and is analytically tractable. We apply the multivariate extension of the stochastic volatility model of Barndorff-Nielsen and Shephard [3], analysed in detail by Pigorsch and Stelzer [21]. The mean-reverting features of our spot model require a significant extension of their analysis.

In this paper we derive the forward dynamics using a no-arbitrage pricing. Despite the rather general nature of our spot model, the dynamics of the forward prices is analytically computable. It turns out that the implied forward curves can be in contango and backwardation, as well as having humps. As has been pointed out by Geman [12], hump-shaped forward curves have been observed in for instance the oil market. Due to the flexibility of the multivariate model, even an oscillation of the forward price curve can be achieved. As an implication of the stationary properties of the spot model, the forward prices in the long-end of the forward curve (far until maturity) will move deterministically. The Samuelson effect can be identified in the forward dynamics as well.

By using Fourier methods, options on spreads between different forward contracts can be represented as integrals which can be computed efficiently. Spread options are traded in various energy markets, mostly over-the-counter. However, such options are also used in valuation of new power plant projects and the construction of interconnecting pipelines between different markets. In fact, the construction of a new pipeline connecting two markets can be viewed as a long term spread option. On the other hand, the value of a gas-fired power plant can be represented as a spread between electricity and gas (so-called spark spread).

The paper is organized as follows. Section 2 recalls the spot model proposed in Benth and Vos [7]. Next, in Section 3, the implied multivariate forward dynamics are derived and properties of the forward curve are analysed. Methods based on the Fourier transform are applied to cross-commodity option pricing in Section 4, including special attention to spread options. Finally, in Section 5, we conclude.

2 A cross-commodity energy spot price model with stochastic volatility

In this section we recall briefly the main aspects of the spot model with stochastic volatility for cross-commodity energy markets introduced in Benth and Vos [7]. We suppose that we are given a complete filtered probability space $(\Omega, \mathcal{F}, P)$ equipped with the filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions (see *e.g.* Protter [22]).

Assume that $m \leq n \in \mathbb{N}$, and for $d \in \mathbb{N}$, consider the d-dimensional spot price dynamics as a combination of a seasonality function Λ, stochastic processes $\{Y_i\}_{i=1}^m$ modeling spikes and a stochastic process X modeling the "normal" variations of the price evolution. Here, the seasonality and the stochastic processes X and $\{Y_i\}_{i=1}^m$ are all d-dimensional. More precisely, we define the spot price dynamics of d energy commodities as follows:

$$S(t) = \Lambda(t) \cdot \exp\left(X(t) + \sum_{i=1}^{m} Y_i(t)\right) . \tag{2.1}$$

Here, '$\cdot$' denotes pointwise multiplication, and the seasonality Λ is supposed to be a deterministic bounded measurable function. The stochastic processes $\{Y_i\}_{i=1}^m$ are d-dimensional Ornstein-Uhlenbeck processes driven by vector valued subordinators $\{L_i\}_{i=1}^m$, that is, Lévy processeses which are increasing in each of its coordinates (see Barndorff-Nielsen et al. [2]).

$$dY_i(t) = (\mu_i + B_i Y_i(t))\, dt + \eta_i\, dL_i(t)\,, \tag{2.2}$$

where $\{\mu_i\}_{i=1}^m$ are vectors in $\mathbb{R}^d$. Furthermore, $\{B_i\}_{i=1}^m$ and $\{\eta_i\}_{i=1}^m$ are elements of $GL_d(\mathbb{R})$, the group of $d \times d$ matrices which are invertible. The entries of η_i do not necessarily have to be positive, so although L_i are subordinators the process Y_i may exhibit negative jumps. In electricity markets, say, negative spikes are observed.

The "normal variations" process X is an extension of the Barndorff-Nielsen and Shephard [3] stochastic volatility (BNS SV) model into the multidimensional Ornstein-Uhlenbeck setting. The stochastic process X is defined by the following SDE

$$dX(t) = AX(t)\,dt + \Sigma(t)^{1/2}\,dW(t)\,, \tag{2.3}$$

where A is a matrix in $GL_d(\mathbb{R})$ and W is a standard d-dimensional Brownian motion in $\mathbb{R}^d$. The square of the volatility $\Sigma(t)$ is chosen to be a matrix valued stochastic process. More precisely, the stochastic volatility $\Sigma(t)$ is a superposition of positive definite matrix valued Ornstein-Uhlenbeck processes as introduced in Barndorff-Nielsen and Stelzer [4],

$$\Sigma(t) = \sum_{j=1}^{n} \omega_j Z_j(t)\,, \tag{2.4}$$

with

$$dZ_j(t) = (C_j Z_j(t) + Z_j(t) C_j^T)dt + d\widetilde{L}_j(t)\,, \tag{2.5}$$

and the ω_j's are positive weights summing up to 1. Moreover, for $j = 1, \ldots, n$, $C_j \in GL_d(\mathbb{R})$ and $\widetilde{L}_j$ are independent matrix valued subordinators, that is, independent increment processes with values in $\mathbb{S}_d^+$, the positive definite cone of symmetric $d \times d$ matrices. Naturally, $\widetilde{L}_j$ are independent of W for $j = 1, \ldots, n$, and we suppose for convenience that the subordinators are driftless. In order to have the Itô integrals in (2.3) well-defined, we suppose that

$$P\left(\int_0^{\widetilde{T}} tr(\Sigma(t))\,dt < \infty\right) = 1\,. \tag{2.6}$$

Here, $\widetilde{T} < \infty$ is some finite horizon time for our energy markets, and tr is the trace operator on matrices. We assume that the eigenvalues of C_j have negative real parts, a necessary condition for ensuring stationarity of the Z_j's. We denote by $\nu_{\widetilde{L}_j}$ the Lévy measure of $\widetilde{L}_j$, $j = 1, \ldots, n$.

The processes X, Y_i are Ornstein-Uhlenbeck processes. Applying the multi-dimensional Itô Formula (see Ikeda and Watanabe [16]) yields the following explicit dynamics: for $0 \le s \le t$,

$$X(t) = \mathbf{e}^{A(t-s)} X(s) + \int_s^t \mathbf{e}^{A(t-u)} \Sigma(u)^{1/2}\,dW(u)\,, \tag{2.7}$$

$$Y_i(t) = \mathbf{e}^{B_i(t-s)} Y_i(s) + B_i^{-1}(I - \mathbf{e}^{B_i(t-s)})\mu_i + \int_s^t \mathbf{e}^{B_i(t-u)} \eta_i\,dL_i(u)\,, \tag{2.8}$$

for $i = 1, \ldots, m$. The matrix exponentials are defined as usual as $\mathbf{e}^A := I + \sum_{i=1}^{\infty} \frac{A^n}{n!}$.

According to Barndorff-Nielsen and Stelzer [4], Sect. 4, the solution of $Z_j(t)$, $j = 1, \ldots, n$, is given by

$$Z_j(t) = \mathbf{e}^{C_j(t-s)} Z_j(s) \mathbf{e}^{C_j^T(t-s)} + \int_s^t \mathbf{e}^{C_j(t-u)} \, d\widetilde{L}_j(u) \mathbf{e}^{C_j^T(t-u)} \,. \tag{2.9}$$

The matrix-valued stochastic integral in the second term of $Z_j(t)$ is understood as follows: let $M_d(\mathbb{R})$ be the space of real $d \times d$ matrices. For two $M_d(\mathbb{R})$-valued bounded and measurable functions $E(u)$ and $F(u)$ on $[t, s]$, the notation $\int_s^t E(u) \, d\widetilde{L}(u) F(u)$ means the matrix $G(s,t) \in M_d(\mathbb{R})$ with coordinates defined by

$$G_{ij}(s,t) = \sum_{k=1}^{d} \sum_{l=1}^{d} \int_s^t E_{ik}(u) F_{lj}(u) \, d\widetilde{L}_{kl}(u) \,.$$

Here, $\widetilde{L}$ is the generic notation for some $\widetilde{L}_j$. We remark that since $\widetilde{L}_j$ are supposed to be RCLL, the processes Z_j also are RCLL.

In energy markets like gas and electricity it is often observed that a spike and an increase in volatility occur at the same time. This is known as the inverse leverage effect. To model this phenomen we take the vector valued subordinators L_i driving the processes Y_i, $i = 1, \ldots, m$, as the diagonal entries of the m first matrix valued subordinators $\widetilde{L}_j$, $j = 1, \ldots, m$. If one of the off-diagonal elements jumps, also the diagonal element has to jump in order to keep the volatility process $\Sigma(t)$ in the positive definite cone $\mathbb{S}_d^+$. Such a modelling choice ensures that the volatility jumps simultaneously with a spike in the spot price process. Since $n \geq m \in \mathbb{N}$, and the volatility process is a weighted sum of n different volatility processes, there are still $n - m$ volatility processes Z_j, $j = m+1, \ldots, n$ which can be freely chosen.

By turning off the processes Y_i (choose $\mu_i = \eta_i = 0$ and $B_i = 0$ for all i), we obtain a multivariate extension of the Schwartz model with stochastic volatility and stock-price dynamics:

$$S(t) = \Lambda(t) \cdot \exp(X(t)) \tag{2.10}$$

where $X(t)$ is defined in (2.3). The Schwartz model with constant volatility is a mean reversion process proposed by Schwartz [24] for spot price dynamics in commodity markets like oil.

To ensure solutions to the SDE's (2.2) and (2.3) we impose the following log integrability conditions on the subordinators: for $j = 1, \ldots, n$, it holds that

$$\mathbb{E}\left[\log^+ ||\widetilde{L}_j(1)||\right] < \infty \,, \tag{2.11}$$

where $\log^+(x)$ is defined as $\max(\log(x), 0)$. We use the Frobenius norm for matrices, $\|A\| = tr(A^T A)^{1/2}$, $A \in M_d(\mathbb{R})$.

For a detailed analysis of this spot price model for cross-commodity energy markets, we refer to Benth and Vos [7].

3 Forward pricing

In commodity markets, forward contracts are commonly traded on exchanges, including power, gas, oil, coal, etc. In this Section we derive the forward price dynamics based on the multivariate spot price model (2.1).

Appealing to general arbitrage theory, we define the forward price $F(t,\tau)$ at time t for contracts delivering the energy commodity at time τ by (see e.g. Duffie [9])

$$F(t,\tau) = \mathbb{E}_Q\left[S(\tau) \,|\, \mathcal{F}_t\right] \,, \tag{3.1}$$

where Q is a risk-neutral probability measure. This definition is valid as long as $S(\tau) \in L^1(Q)$. Below we give sufficient conditions ensuring integrability of the spot price with respect to a parametric class of pricing meausres Q. Since the spot price is an adapted process, we obtain the well-known convergence of spot and forward prices at maturity, *i.e.*,

$$F(\tau,\tau) = S(\tau)\,.$$

It is worth noticing that in some energy markets the forward contracts deliver the underlying commodity over a period rather than at a fixed maturity time τ. This includes gas and electricity, but also more exotic markets like temperature. In these markets, the forward prices can be represented as some functional of $F(t,\tau)$, usually the average of $F(t,\tau)$ over τ, taken over the delivery period of the forward contract. We will not consider this situation here, however the calculations can be easily adjusted to take this into account (see for example Benth *et al.* [6] for a discussion).

The stochastic volatility model we are discussing gives rise to an incomplete market, and hence there exists a continuum of equivalent martingale measures Q that can be used for pricing. Moreover, in energy markets, the underlying spot is in general not tradeable, due to for example high storage costs, illiquidity and other frictions like transportation for delivery. In the extreme case of electricity, it is impossible to trade the underlying spot by the very nature of the commodity. Hence, the classical buy-and-hold hedging argument to pin down a forward price fails. As a result, *all* equivalent measures $Q \sim P$ may be chosen as pricing measures since the underlying spot is not directly tradeable. In our considerations., we do not require the martingale property under Q for discounted spot prices. We refer to Benth *et al.* [6] for more on this.

3.1 A class of equivalent probabilities

A convenient way to define a parametric class of risk-neutral probabilities for Lévy-based models is the Esscher transform (see Benth *et al.* [6] for applications of the Esscher transform in energy markets). Before introducing the measure transform, we need to introduce some notation and state some conditions: for $V \in \mathbb{S}_d^+$ we let $\phi_{\widetilde{L}_j}(V)$ be the cumulant function of $\widetilde{L}_j(1)$, that is,

$$\phi_{\widetilde{L}_j}(V) = \ln \mathbb{E}\left[\exp\left(\mathrm{i}tr(V\widetilde{L}_j(1))\right)\right] . \tag{3.2}$$

The Esscher transform is defined via the logarithmic moment generating functions of $\widetilde{L}_j$, and for this purpose we need to have certain exponential moments exisiting for $\widetilde{L}_j$. Let $\Theta_j \in \mathbb{S}_d^+$, and suppose that $\phi_{\widetilde{L}_j}(-\mathrm{i}\Theta_j)$ is well-defined. We have that

$$\phi_{\widetilde{L}_j}(-\mathrm{i}\Theta_j) = \int_{\mathbb{S}_d^+} \left\{\mathbf{e}^{tr(\Theta_j U)} - 1\right\} \nu_{\widetilde{L}_j}(dU) \,,$$

and therefore, $\phi_{\widetilde{L}_j}(-\mathrm{i}\Theta_j)$ is well-defined as long as

$$\int_{\mathbb{S}_d^+} \left\{\mathbf{e}^{tr(\Theta_j U)} - 1\right\} \nu_{\widetilde{L}_j}(dU) < \infty \,. \tag{3.3}$$

Note that for $U, V \in \mathbb{S}_d^+$, $tr(UV) = \langle U, V\rangle$, the inner product associated with the Frobenius matrix norm $\|A\| := tr(A^T A)^{1/2}$. Hence, we have the inequality $|tr(UV)| \leq \|U\|\|V\|$. Thus, a sufficient condition for (3.3) to hold is that

$$\int_{\mathbb{S}_d^+} \mathbf{e}^{|tr(\Theta_j U)|} \, \nu_{\widetilde{L}_j}(dU) \leq \int_{\mathbb{S}_d^+} \mathbf{e}^{\|\Theta_j\|\|U\|} \, \nu_{\widetilde{L}_j}(dU) < \infty \,.$$

Throughout this paper we suppose that there exists a constant $c_j > 0$ such that the following exponential integrability condition holds for $\nu_{\widetilde{L}_j}$:

$$\int_{\mathbb{S}_d^+} \mathbf{e}^{c_j\|U\|} \, \nu_{\widetilde{L}_j}(dU) < \infty \,, \tag{3.4}$$

for $j = 1, \ldots, n$. This condition implies that $\phi_{\widetilde{L}_j}(-\mathrm{i}\Theta_j)$ is well-defined for all $\Theta_j \in \mathbb{S}_d^+$ such that $\|\Theta_j\| \leq c_j$.

We move on to define the equivalent probability measure Q. For $\Theta_j \in \mathbb{S}_d^+$, such that $\|\Theta_j\| \leq c_j$, define the processes

$$\mathcal{V}_j(t) = \exp\left(tr(\Theta_j\widetilde{L}_j(t)) - \phi_{\widetilde{L}}^j(-\mathrm{i}\Theta_j)t\right) \,, \tag{3.5}$$

for $j = 1, \ldots, n$ and $t \leq \widetilde{T}$. Here we recall $\widetilde{T}$ to be a finite time horizon of the market for which all delivery times τ of interest are included. Note that $V_j(t)$ are martingales for $j = 1, \ldots, m$: in fact, by the exponential moment condition in (3.4) we find that

$$\mathbb{E}[\mathcal{V}_j(t)] = 1\,,$$

for every $j = 1, \ldots, n$. For a vector $\theta_0 \in \mathbb{R}^d$, introduce the process

$$\mathcal{V}_0(t) = \exp\left(-\int_0^t \theta_0^T \Sigma^{-1/2}(s)\,dW(s) - \frac{1}{2}\theta_0^T \int_0^t \Sigma^{-1}(s)\theta_0\,ds\right)\,. \tag{3.6}$$

We have the following Lemma:

Lemma 3.1. *For all $\theta_0 \in \mathbb{R}^d$, the process $\mathcal{V}_0(t)$ for $t \leq \widetilde{T}$ is a martingale.*

Proof. We show that the Novikov condition holds. From (2.9) we have for every $j = 1, \ldots, n$ and any $x \in \mathbb{R}^d$

$$\begin{aligned} x^T Z_j(t) x &= x^T \mathbf{e}^{C_j t} Z_j(0) \mathbf{e}^{C_j^T t} x + x^T \int_0^t \mathbf{e}^{C_j(t-u)}\,d\widetilde{L}_j(u) \mathbf{e}^{C_j^T(t-u)} x \\ &\geq x^T \mathbf{e}^{C_j(t-s)} Z_j(s) \mathbf{e}^{C_j^T(t-s)} x \end{aligned}$$

by positive definiteness of the stochastic integral term. Hence,

$$\Sigma(t) = \sum_{j=1}^n \omega_j Z_j(t) \geq \sum_{j=1}^n \omega_j \mathbf{e}^{C_j(t-s)} Z_j(s) \mathbf{e}^{C_j^T(t-s)} > 0\,.$$

But then, from linear algrebra on positive definite matrices,

$$\Sigma^{-1}(t) \leq \left(\sum_{j=1}^n \omega_j \mathbf{e}^{-C_j t} Z_j^{-1}(0) \mathbf{e}^{-C_j^T t}\right)^{-1}\,,$$

which means in particular

$$\theta_0^T \Sigma^{-1}(t)\theta_0 \leq \theta_0^T \left(\sum_{j=1}^n \omega_j \mathbf{e}^{-C_j t} Z_j^{-1}(0) \mathbf{e}^{-C_j^T t}\right)^{-1} \theta_0\,.$$

As the right-hand side is a continuous function in t on $[0, \widetilde{T}]$, it follows that

$$\mathbb{E}\left[\exp\left(\frac{1}{2}\int_0^{\widetilde{T}} \theta_0^T \Sigma^{-1}(t)\theta_0\,dt\right)\right]$$

$$\leq \mathbb{E}\left[\exp\left(\frac{1}{2}\int_0^{\widetilde{T}} \theta_0^T \left(\sum_{j=1}^n \omega_j \mathbf{e}^{-C_j t} Z_j^{-1}(0)\mathbf{e}^{-C_j^T t}\right)^{-1} \theta_0 \, dt\right)\right] < \infty \, .$$

Hence, by Novikov's condition, it follows from the Girsanov Theorem that $\mathcal{V}_0(t)$ is a martingale. □

Thus, the process

$$\mathcal{V}(t) = \mathcal{V}_0(t) \times \mathcal{V}_1(t) \times \cdots \times \mathcal{V}_n(t) \, , \tag{3.7}$$

becomes a martingale for $t \leq \widetilde{T}$ and is the density process of a probability measure Q equivalent with P, that is,

$$\frac{dQ}{dP}\Big|_{\mathcal{F}_t} = \mathcal{V}(t) \, . \tag{3.8}$$

From Girsanov's Theorem we find that

$$d\widehat{W}(t) = dW(t) - \Sigma^{-1/2}(t)\theta_0 \, dt \, , \tag{3.9}$$

is an $\mathbb{R}^d$-valued Brownian motion with respect to Q on $t \in [0, \widetilde{T}]$. Furthermore, $\widetilde{L}_j(t)$ is a matrix-valued subordinator with respect to Q, with characteristics stated in the following Lemma:

Lemma 3.2. *Assume $\Theta_j \in \mathbb{S}_d^+$ such that $\|\Theta_j\| \leq c_j$ for $j = 1, \ldots, n$. Then $\widetilde{L}_j(t)$ are subordinators under Q defined in* (3.8) *having Lévy measure with respect to Q given by*

$$\nu^Q_{\widetilde{L}_j}(dU) = \exp(tr(\Theta_j U))\nu_{\widetilde{L}_j}(dU) \, ,$$

for $j = 1, \ldots, n$.

Proof. First we prove that $\widetilde{L}_j(t)$ is a matrix-valued subordinator under Q. Consider its conditional cumulant function with respect to Q, $\widetilde{\phi}^{(s,t)}_{\widetilde{L}_j}(V)$: for $0 \leq s \leq t$ and using Bayes' Formula for conditional expectations (see Karatzas and Shreve [18])

$$\begin{aligned}
\widetilde{\phi}^{(s,t)}_{\widetilde{L}_j}(V) &= \ln \mathbb{E}_Q \left[\exp\left(\mathrm{i}tr(V(\widetilde{L}_j(t) - \widetilde{L}_j(s)))\right) \mid \mathcal{F}_s\right] \\
&= \ln \mathbb{E}\left[\exp\left(\mathrm{i}tr(V(\widetilde{L}_j(t) - \widetilde{L}_j(s)))\right) \frac{\mathcal{V}(t)}{\mathcal{V}(s)} \mid \mathcal{F}_s\right] \\
&= \ln \mathbb{E}\left[\exp\left(\mathrm{i}tr((V - \mathrm{i}\Theta_j)\widetilde{L}_j(1))\right) \mid \mathcal{F}_s\right] - \phi_{\widetilde{L}_j}(-\mathrm{i}\Theta_j)(t-s) \\
&= \ln \mathbb{E}\left[\exp\left(\mathrm{i}tr((V - \mathrm{i}\Theta_j)\widetilde{L}_j(1))\right)\right] - \phi_{\widetilde{L}_j}(-\mathrm{i}\Theta_j)(t-s) \\
&= \phi_{\widetilde{L}_j}(V - \mathrm{i}\Theta_j)(t-s) - \phi_{\widetilde{L}_j}(-\mathrm{i}\Theta_j)(t-s) \, .
\end{aligned}$$

In the second to the last equality we used the independent increment property of $\widetilde{L}_j(t)$. This proves that the increment $\widetilde{L}_j(t) - \widetilde{L}_j(s)$ is stationary and independent of $\mathcal{F}_s$, hence a Lévy process with respect to the probability Q. Moreover, $\widetilde{L}_j(t)$ has values in $\mathcal{S}_d^+$, and therefore it is a subordinator under Q. From the above calculation we find its cumulant under Q to be

$$\begin{aligned}
\widetilde{\phi}_{\widetilde{L}_j} &:= \ln \mathbb{E}_Q \left[\exp\left(\mathrm{i}tr(V\widetilde{L}_j(1))\right)\right] \\
&= \phi_{\widetilde{L}_j}(V - \mathrm{i}\Theta_j) - \phi_{\widetilde{L}_j}(-\mathrm{i}\Theta_j) \\
&= \int_{\mathbb{S}_d^+} \left\{\mathbf{e}^{\mathrm{i}tr((V-\mathrm{i}\Theta_j)U)} - 1\right\} \nu_{\widetilde{L}_j}(dU) - \int_{\mathbb{S}_d^+} \left\{\mathbf{e}^{\mathrm{i}tr((-\mathrm{i}\Theta_j)U)} - 1\right\} \nu_{\widetilde{L}_j}(dU) \\
&= \int_{\mathbb{S}_d^+} \left\{\mathbf{e}^{\mathrm{i}tr(VU)} - 1\right\} \mathbf{e}^{tr(\Theta_j U)} \nu_{\widetilde{L}_j}(dU)\,.
\end{aligned}$$

Hence, the Lemma follows. □

Since a subordinator is a pure-jump process, we must have that $\widetilde{L}_j$ for $j = 1, \ldots, m$ are independent of $\widehat{W}$ with respect to Q, since a Brownian motion has continuous paths.

The parameters θ_0 and Θ_j, $j = 1, \ldots, n$ may be referred to as *the market prices of risk*, extending the similar notion in the univariate case (see Benth *et al.* [6]). Note that the Esscher transform gives an exponential tilting of the Lévy measure of the matrix-valued subordinators $\widetilde{L}_j$. One effect of this is that the probabilities for large jumps are re-scaled, and we may get more or less pronounced large jumps under Q.

The dynamics of $X(t)$ under Q is given by

$$\begin{aligned}
dX(t) &= AX(t) + \Sigma^{1/2}(t)\, \left(d\widehat{W}(t) + \Sigma^{-1/2}(t)\theta_0\, dt\right) \\
&= (\theta_0 + AX(t))\, dt + \Sigma^{1/2}(t)\, d\widehat{W}(t)\,.
\end{aligned} \tag{3.10}$$

Thus, under Q, the mean-reversion level is shifted from 0 to θ_0. If $e_k^T\theta_0 > 0$ for a $k = 1, \ldots, d$ and e_k being the kth canonical unit vector of $\mathbb{R}^d$, then the base component of the kth commodity mean-reverts towards a higher level under Q than under P, implying that the market assesses the base component as being more risky under the pricing measure Q. A negative market price of risk $e_k^T\theta_0$ will imply less risk loading on the kth base component. The dynamics of Y_i and Z_j are changed in a similar fashion. We have for $i = 1, \ldots, m$

$$\begin{aligned}
dY_i(t) &= (\mu_i + B_iY_i(t))\, dt + \eta_i\, dL_i(t) \\
&= (\mu_i + \eta_i\mathbb{E}_Q[L(1)] + B_iY_i(t))\, dt + \eta_i\, dL_i^Q(t)\,,
\end{aligned} \tag{3.11}$$

where $dL_i^Q(t) \triangleq dL_i(t) - \mathbb{E}_Q[L_i(1)]\, dt$ is a Q-martingale. Hence, the process Y_i varies around the level $\mu_i + \eta_i\mathbb{E}_Q[L_i(1)]$ under Q, whereas the level is $\mu_i + \eta_i\mathbb{E}[L_i(1)]$ under P. Thus, by appropriately choosing Θ_i we can obtain a higher or lower mean-reversion level, implying a higher or lower risk loading on the spike processes Y_i under Q. Similar considerations hold for the volatility processes Z_j. We remark in passing that the market prices of risk $\theta_0, \Theta_1, \ldots, \Theta_n$ will implicitly model the risk premium in the market, being the difference between the forward price and the prediced spot at delivery.

3.2 Analysis of forward prices

Before we derive the forward price, we need to introduce some notation and prove an auxiliary result. To this end, let J_d be the linear operator that maps a vector $v \in \mathbb{R}^d$ to a symmetric $d \times d$-matrix $J_d(v)$, consisting of zeros except on the diagonal, which is equal to v. On the other hand, diag is a linear operator mapping a matrix into a vector, where the vector is the diagonal of the matrix. The family of linear operators $\mathcal{C}_j(t)$ for $t \in [0, \widetilde{T}]$ are defined as

$$\mathcal{C}_j(t) : X \mapsto \omega_j \left[(\mathbf{C_j} - \mathbf{A})^{-1} \left(e^{C_j t} X e^{C_j^T t} - e^{At} X e^{A^T t} \right) \right] , \tag{3.12}$$

for $j = 1, \ldots, n$. For A being an $n \times n$-matrix, we denote the operator $\mathbf{A}$ associated with the matrix A as $\mathbf{A} : X \mapsto AX + XA^T$. This operator can be represented as $\mathrm{vec}^{-1} \circ ((A \otimes I_n) + (I_n \otimes A)) \circ \mathrm{vec}$, with I_n being the $n \times n$ identity matrix and vec meaning the operator which stacks the coloumns of a matrix into a vector. Its inverse is denoted by $\mathbf{A}^{-1}$, which exists whenever $I_n \otimes A + A \otimes I_n$ is invertible. In this case, we can represent $\mathbf{A}^{-1}$ by $\mathrm{vec}^{-1} \circ ((A \otimes I_n) + (I_n \otimes A))^{-1} \circ \mathrm{vec}$. Remark that $A \otimes I_n + I_n \otimes A$ is equal to the Kronecker sum of the matrix A with itself.

The following auxiliary result is useful in deriving the forward prices, and is proven in Benth and Vos [7].

Lemma 3.3. *Define* $f(s,t) := \int_s^t e^{A(t-u)} \Sigma(u) e^{A^T(t-u)} du$. *Assume for* $j = 1, \ldots, n$ *that* A *and* C_j *commute and* $\mathbf{A} - \mathbf{C}_j$ *are invertible. Then it holds*

$$f(s,t) = \sum_{j=1}^{n} \mathcal{C}_j(t-s) Z_j(s) + \int_s^t \mathcal{C}_j(t-v) d\widetilde{L}_j(v) \,,$$

for $0 \le s \le t$.

Proof. The proof of this result is found in Benth and Vos [7]. We include it here for the convenience of the reader. Using (2.9) and the assumption that A and C_j commute for $j = 1, \ldots, n$ it holds

$$f(s,t) = \int_s^t \mathbf{e}^{A(t-u)} \sum_{j=1}^{n} \omega_j \left(\mathbf{e}^{C_j(u-s)} Z_j(s) \mathbf{e}^{C_j^T(u-s)} + \int_s^u \mathbf{e}^{C_j(u-v)} d\widetilde{L}_j(v) \mathbf{e}^{C_j^T(u-v)} \right) \mathbf{e}^{A^T(t-u)} du$$

$$
\begin{aligned}
&= \sum_{j=1}^{n} \omega_j \int_s^t \mathbf{e}^{(C_j - A)u} \mathbf{e}^{At - C_j s} \left(Z_j(s) + \int_s^u \mathbf{e}^{-C_j v} d\widetilde{L}_j(v) \mathbf{e}^{-C_j^T v} \right) \mathbf{e}^{A^T t - C_j^T s} \mathbf{e}^{(C_j - A)^T u} du \\
&= \sum_{j=1}^{n} \omega_j (\mathbf{C_j} - \mathbf{A})^{-1} \left(\mathbf{e}^{C_j(t-s)} Z_j(s) e^{C_j^T(t-s)} - \mathbf{e}^{A(t-s)} Z_j(s) \mathbf{e}^{A^T(t-s)} \right) \\
&\quad + \int_s^t \int_s^u \left\{ \mathbf{e}^{(C_j - A)u} \mathbf{e}^{At} \mathbf{e}^{-C_j v} d\widetilde{L}_j(v) \mathbf{e}^{-C_j^T v} \mathbf{e}^{A^T t} \mathbf{e}^{(C_j - A)^T u} \right\} du \,.
\end{aligned}
$$

The last integral is intepreted as *first* integrating with respect to $d\widetilde{L}_j(v)$, and *next* integrating the obtained expression with respect to du. But, by spelling out the integrals in terms of sums, using the definition of the $d\widetilde{L}_j(v)$ integrals, and invoking the stochastic Fubini theorem (see Protter [22]), we get

$$
\begin{aligned}
&\int_s^t \int_s^u \left\{ \mathbf{e}^{(C_j - A)u} \mathbf{e}^{At} \mathbf{e}^{-C_j v} d\widetilde{L}_j(v) \mathbf{e}^{-C_j^T v} \mathbf{e}^{A^T t} \mathbf{e}^{(C_j - A)^T u} \right\} du \\
&\qquad = \int_s^t \int_v^t \left\{ \mathbf{e}^{(C_j - A)u} \mathbf{e}^{At} \mathbf{e}^{-C_j v} d\widetilde{L}_j(v) \mathbf{e}^{-C_j^T v} \mathbf{e}^{A^T t} \mathbf{e}^{(C_j - A)^T u} \right\} du \,.
\end{aligned}
$$

Here, the right hand side is intepreted as *first* integrating with respect to du, treating $d\widetilde{L}_j(v)$ as a matrix and not a differential, and *next* integrating with respect to $d\widetilde{L}_j(v)$ the obtained expression. Hence, we find

$$
\begin{aligned}
f(s,t) &= \sum_{j=1}^{n} \mathcal{C}_j(t-s) Z_j(s) \\
&\quad + (\mathbf{C_j} - \mathbf{A})^{-1} \left(\int_s^t \mathbf{e}^{C_j(t-v)} d\widetilde{L}_j(v) \mathbf{e}^{C_j^T(t-v)} - \int_s^t \mathbf{e}^{A(t-v)} d\widetilde{L}_j(v) \mathbf{e}^{A^T(t-v)} \right) .
\end{aligned}
$$

The Lemma follows. □

By $\mathcal{C}_j^*(u)$ we mean the adjoint operator of $\mathcal{C}_j(u)$. Since $\mathcal{C}_j(u)$ is a linear operator on $d \times d$-matrices, one can represent it via a $d^2 \times d^2$-matrix $\mathcal{K}_j(u)$ by $\mathcal{C}_j(u) = \text{vec}^{-1} \circ \mathcal{K}_j(u) \circ \text{vec}$. Hence, the adjoint $\mathcal{C}_j^*(u)$ has the representation $\mathcal{C}_j^*(u) = \text{vec}^{-1} \circ \mathcal{K}_j^T(u) \circ \text{vec}$.

We are now in the position to state the forward price.

Proposition 3.4. *For $k = 1, \ldots, d$, suppose Θ_j are such that*

$$
\sup_{u \in [0, \widetilde{T}]} \| \frac{1}{2} \mathcal{C}_j^*(u)(e_k e_k^T) \| + \|\Theta_j\| \leq c_j
$$

for $j = 1, \ldots, n$, and

$$\sup_{u\in[0,\widetilde{T}]} \|\frac{1}{2}\mathcal{C}_i^*(u)(e_k e_k^T) + J_d(e_k^T \boldsymbol{e}^{B_i u}\eta_i)\| + \|\Theta_i\| \leq c_i$$

for $i = 1, \ldots, m$. Assume for $j = 1, \ldots, n$ that A and C_j commute and $\mathbf{A} - \mathbf{C}_j$ are invertible. Then the forward price at time $t \geq 0$ of a contract delivering the d spots $S(\tau)$ at time $\tau \geq t$ is

$$\begin{aligned} F(t,\tau) = \Lambda(\tau)\cdot\exp\Bigg(&\boldsymbol{e}^{A(\tau-t)}X(t) + \sum_{i=1}^m \boldsymbol{e}^{B_i(\tau-t)}Y_i(t) + A^{-1}(I-\boldsymbol{e}^{A(\tau-t)})\theta_0 \\ &+ \sum_{i=1}^m B_i^{-1}(I-\boldsymbol{e}^{B_i(\tau-t)})\mu_i + \frac{1}{2}\mathrm{diag}\{\sum_{j=1}^n \mathcal{C}_j(\tau-t)Z_j(t)\}\Bigg)\cdot\Psi(\tau-t)\,, \end{aligned} \tag{3.13}$$

where the kth coordinate of $\Psi(s) \in \mathbb{R}^d$ for $0 \leq s \leq \widetilde{T}$ is

$$\begin{aligned} \ln\Psi_k(s) = &\sum_{j=1}^n \int_0^s \left\{\phi_{\widetilde{L}_j}\left(-\frac{1}{2}\mathrm{i}\mathcal{C}_j^*(u)(e_k e_k^T) - \mathrm{i}\Theta_j\right) - \phi_{\widetilde{L}_j}(-\mathrm{i}\Theta_j)\right\} du \\ &+ \sum_{i=1}^m \int_0^s \Bigg\{\phi_{\widetilde{L}_i}\left(-\frac{1}{2}\mathrm{i}\mathcal{C}_i^*(u)(e_k e_k^T) - \mathrm{i}J_d(e_k^T\boldsymbol{e}^{B_i u}\eta_i) - \mathrm{i}\Theta_i\right) \\ &\qquad - \phi_{\widetilde{L}_i}(-\frac{1}{2}\mathrm{i}\mathcal{C}_i^*(u)(e_k e_k^T) - \mathrm{i}\Theta_i)\Bigg\}\, du\,, \end{aligned}$$

for $k = 1, \ldots, d$.

Proof. For simplicity, we let $m = n = 1$ and defer the subscripts with respect to i and j. From (2.7) and (2.8) along with the definition of the measure Q, we have

$$\begin{aligned} X(\tau) &= \mathbf{e}^{A(\tau-t)}X(t) + \int_t^\tau \mathbf{e}^{A(\tau-u)}\Sigma^{1/2}(u)\,dW(u) \\ &= \mathbf{e}^{A(\tau-t)}X(t) + \int_t^\tau \mathbf{e}^{A(\tau-u)}\theta_0\,du + \int_t^\tau \mathbf{e}^{A(\tau-u)}\Sigma^{1/2}(u)\,d\widehat{W}(u) \\ &= \mathbf{e}^{A(\tau-t)}X(t) + A^{-1}(I-\mathbf{e}^{A(\tau-t)})\theta_0 + \int_t^\tau \mathbf{e}^{A(\tau-u)}\Sigma^{1/2}(u)\,d\widehat{W}(u)\,, \end{aligned}$$

and

$$Y(\tau) = \mathbf{e}^{B(\tau-t)}Y(t) + B^{-1}(I-\mathbf{e}^{B(\tau-t)}\mu + \int_t^\tau \mathbf{e}^{B(\tau-u)}\eta\,dL(u)\,.$$

Hence, using the $\mathcal{F}_t$-adaptedness of $X(t)$ and $Y(t)$, we find

$$\begin{aligned}F(t,\tau) &= \Lambda(\tau)\cdot\mathbb{E}_Q\left[\exp(X(\tau)+Y(\tau))\,|\,\mathcal{F}_t\right]\\ &= \Lambda(\tau)\cdot\exp\left(\mathbf{e}^{A(\tau-t)}X(t)+\mathbf{e}^{B(\tau-t)}Y(t)+A^{-1}(I-\mathbf{e}^{A(\tau-t)})\theta_0+B^{-1}(I-\mathbf{e}^{B(\tau-t)})\mu\right)\\ &\quad\cdot\; E_Q\left[\exp\left(\int_t^\tau \mathbf{e}^{A(\tau-u)}\Sigma^{1/2}(u)\,d\widehat{W}(u)+\int_t^\tau \mathbf{e}^{B(\tau-u)}\eta\,dL(u)\right)\,|\,\mathcal{F}_t\right]\end{aligned}$$

We consider the expectation in the last equality, which we denote by $\widehat{F}(t,\tau)$. Let $\mathcal{G}_{t,\tau}$ be the σ-algebra generated by $\mathcal{F}_t$ and $\widetilde{L}(u)$ for $t\le u\le\tau$. Recalling that under Q, $\widehat{W}$ and $\widetilde{L}$ are independent, we find from the tower property of the conditional expectation operator

$$\begin{aligned}\widehat{F}(t,\tau) &= \mathbb{E}_Q\left[\mathbb{E}_Q\left[\exp\left(\int_t^\tau \mathbf{e}^{A(\tau-s)}\Sigma^{1/2}(s)\,d\widehat{W}(s)+\int_t^\tau \mathbf{e}^{B(\tau-s)}\eta\,dL(s)\right)\,|\,\mathcal{G}_{t,\tau}\right]\,|\,\mathcal{F}_t\right]\\ &= \mathbb{E}_Q\left[\exp\left(\int_t^\tau \mathbf{e}^{B(\tau-s)}\eta\,dL(s)\right)\cdot\mathbb{E}_Q\left[\exp\left(\int_t^\tau \mathbf{e}^{A(\tau-s)}\Sigma^{1/2}(s)\,d\widehat{W}(s)\right)\,|\,\mathcal{G}_{t,\tau}\right]\,|\,\mathcal{F}_t\right]\\ &= \mathbb{E}_Q\left[\exp\left(\frac{1}{2}\mathrm{diag}\left[\int_t^\tau \mathbf{e}^{A(\tau-s)}\Sigma(s)\mathbf{e}^{A^T(\tau-s)}\,ds\right]+\int_t^\tau \mathbf{e}^{B(\tau-s)}\eta\,dL(s)\right)\,|\,\mathcal{F}_t\right].\end{aligned}$$

In the second equality we used that L is measurable with respect to $\mathcal{G}_{t,\tau}$, while in the last equality we applied the facts that the Wiener integral of a deterministic function is independent of $\mathcal{F}_t$ and a Gaussian random variable.

From Lemma 3.3, we find after appealing to the $\mathcal{F}_t$-measurability of $Z(t)$ and the independent increment property of Lévy processes,

$$\begin{aligned}\widehat{F}(t,\tau) &= \mathbb{E}_Q\left[\exp\left(\frac{1}{2}\mathrm{diag}(\mathcal{C}(\tau-t)Z(t))+\frac{1}{2}\mathrm{diag}(\int_t^\tau \mathcal{C}(\tau-u)\,d\widetilde{L}(u))+\int_t^\tau \mathbf{e}^{B(\tau-u)}\eta\,dL(u)\right)\,|\,\mathcal{F}_t\right]\\ &= \exp\left(\frac{1}{2}\mathrm{diag}(\mathcal{C}(\tau-t)Z(t))\right)\\ &\quad\cdot\mathbb{E}_Q\left[\exp\left(\frac{1}{2}\mathrm{diag}(\int_t^\tau \mathcal{C}(\tau-u)\,d\widetilde{L}(u))+\int_t^\tau \mathbf{e}^{B(\tau-u)}\eta\,dL(u)\right)\right].\end{aligned}$$

Let us focus on the expectation above, and denote it by $\Psi(t,\tau)$. It is a vector in $\mathbb{R}^d$, and we look at it componentwise. Note that the kth coordinate of $\mathrm{diag}(\int_t^\tau \mathcal{C}(\tau-u)\,d\widetilde{L}(u))$ can be expressed as $e_k^T\int_t^\tau \mathcal{C}(\tau-u)\,d\widetilde{L}(u)e_k$, while the kth coordinate of $\int_t^\tau \mathbf{e}^{B(\tau-u)}\eta\,dL(u)$ is $e_k^T\int_t^\tau \mathbf{e}^{B(\tau-u)}\eta\,dL(u)$. Hence, from the fundamental relation $w^kUw=tr(ww^kA)$ for a vector w and a matrix U,

$$\begin{aligned}\Psi_k(t,\tau) &= \mathbb{E}_Q\left[\exp\left(\frac{1}{2}e_k^T\int_t^\tau \mathcal{C}(\tau-u)\,d\widetilde{L}(u)e_k+e_k^T\int_t^\tau \mathbf{e}^{B(\tau-u)}\eta\,dL(u)\right)\right]\\ &= \mathbb{E}_Q\left[\exp\left(\mathrm{i}tr(-\frac{1}{2}\mathrm{i}e_ke_k^T\int_t^\tau \mathcal{C}(\tau-u)\,d\widetilde{L}(u))+\int_t^\tau e_k\mathbf{e}^{B(\tau-u)}\eta\,dL(u)\right)\right]\end{aligned}$$

Note that $e_k^T \mathbf{e}^{B(\tau-u)}\eta$ is a d-dimensional vector. It is simple to see that

$$\int_t^\tau e_k^T \mathbf{e}^{B(\tau-u)}\eta \, dL(u) = tr(\int_t^\tau J_d(e_k^T \mathbf{e}^{B(\tau-u)}\eta) \, d\widetilde{L}(u) \, .$$

Hence,

$$\begin{aligned}
\Psi_k(t,\tau) &= \mathbb{E}_Q\left[\exp\left(\mathrm{i}tr(-\frac{1}{2}\mathrm{i}\int_t^\tau e_k e_k^T \mathcal{C}(\tau-u)\, d\widetilde{L}(u)) + \mathrm{i}tr(-\mathrm{i}\int_t^\tau J_d(e_k^T \mathbf{e}^{B(\tau-u)}\eta)\, d\widetilde{L}(u))\right)\right] \\
&= \mathbb{E}\left[\exp\left(\mathrm{i}tr\left(\int_t^\tau \left\{-\frac{1}{2}\mathrm{i}e_k e_k^T \mathcal{C}(\tau-u) - \mathrm{i}J_d(e_k^T \mathbf{e}^{B(\tau-u)}\eta)\right\} d\widetilde{L}(u)\right)\right)\right] \\
&\quad \times \exp\left(-\phi_{\widetilde{L}}(-\mathrm{i}\Theta)\right)
\end{aligned}$$

Next, observe that the stochastic integral can be expressed as

$$\begin{aligned}
&\int_t^\tau \left\{\frac{1}{2}e_k e_k^T \mathcal{C}(\tau-u) + J_d(e_k^T \mathbf{e}^{B(\tau-u)}\eta)\right\} d\widetilde{L}(u) \\
&\quad = \lim_{|\Delta_i|\to 0} \sum_{i=0}^{n-1} \left\{\frac{1}{2}e_k e_k^T \mathcal{C}(\tau-u_i) + J_d(e_k^T \mathbf{e}^{B(\tau-u)}\eta\right\} \Delta\widetilde{L}(u_i) \, ,
\end{aligned}$$

for partitions $t = u_0 < \cdots < u_n = \tau$ with $\Delta_i := \widetilde{L}(u_{i+1}) - \widetilde{L}(u_i)$ and $\Delta_i := u_{i+1} - u_i$. By independence of increments of a Lévy process, and continuity of the exponential function together with Fubini-Tonelli's Theorem, we get

$$\begin{aligned}
&\mathbb{E}\left[\exp\left(\mathrm{i}tr\left(\int_t^\tau \left\{-\frac{1}{2}\mathrm{i}e_k e_k^T \mathcal{C}(\tau-u) - \mathrm{i}J_d(e_k^T \mathbf{e}^{B(\tau-u)}\eta)\right\} d\widetilde{L}(u)\right)\right)\right] \\
&\quad = \lim_{|\Delta_i|\to 0} \prod_{i=1}^{n-1} \mathbb{E}\left[\exp\left(\mathrm{i}tr\left(\left\{-\frac{1}{2}\mathrm{i}e_k e_k^T \mathcal{C}_j(\tau-u_i) - \mathrm{i}J_d(e_k^T \mathbf{e}^{B(\tau-u_i)}\eta)\right\} \Delta\widetilde{L}(u_i)\right)\right)\right] \, .
\end{aligned}$$

Now, the linear operators $\mathcal{C}(\tau-u_i)$ can be represented as $\mathrm{vec}^{-1} \circ \mathcal{K}(\tau-u_i) \circ \mathrm{vec}$ for a matrix $\mathcal{K} \in \mathbb{R}^{d^2\times d^2}$. Hence, since for quadratic matrices $tr(VX) = \mathrm{vec}(V)^T \mathrm{vec}(X)$, we find

$$\begin{aligned}
tr\left((e_k e_k^T)\mathcal{C}(\tau-u_i)\Delta\widetilde{L}(u_i)\right) &= \mathrm{vec}(e_k e_k^T)^T \mathrm{vec}\left(\mathcal{C}(\tau-u_i)\Delta\widetilde{L}(u_i)\right) \\
&= \mathrm{vec}(e_k e_k^T)^T \mathrm{vec}\left(\mathrm{vec}^{-1}\mathcal{K}(\tau-u_i)\mathrm{vec}(\Delta\widetilde{L}(u_i))\right) \\
&= \mathrm{vec}(e_k e_k^T)^T \mathcal{K}(\tau-u_i)\mathrm{vec}(\Delta\widetilde{L}(u_i)) \\
&= \left(\mathcal{K}^T(\tau-u_i)\mathrm{vec}(e_k e_k^T)\right)^T \mathrm{vec}(\Delta\widetilde{L}(u_i)) \\
&= tr\left(\mathrm{vec}^{-1}(\mathcal{K}^T(\tau-u_i)\mathrm{vec}(e_k e_k^T))\Delta\widetilde{L}(u_i)\right)
\end{aligned}$$

$$= tr\left(\mathcal{C}^*(\tau-u_i)(e_ke_k^T)\Delta\widetilde{L}(u_i)\right) .$$

Thus,

$$\mathbb{E}\left[\exp\left(\mathrm{i}tr\left(\left\{-\frac{1}{2}\mathrm{i}e_ke_k^T\mathcal{C}_j(\tau-u_i)-\mathrm{i}J_d(e_k^T\mathbf{e}^{B(\tau-u_i)}\eta)\right\}\Delta\widetilde{L}(u_i)\right)\right)\right]$$
$$=\exp\left(\phi_{\widetilde{L}}\left(-\frac{1}{2}\mathrm{i}\mathcal{C}^*(\tau-u_i)(e_ke_k^T)-\mathrm{i}J_d(e_k\mathbf{e}^{B(\tau-u_i)}\eta)\right)\Delta u_i\right) .$$

Gathering information, we find that

$$\ln\Psi_k(t,\tau)=\int_t^\tau\left\{\phi_{\widetilde{L}}(-\frac{1}{2}\mathrm{i}\mathcal{C}^*(\tau-u)(e_ke_k^T)-\mathrm{i}J_d(e_k^T\mathbf{e}^{B(\tau-u)}\eta)-\mathrm{i}\Theta)-\phi_{\widetilde{L}}(-\mathrm{i}\Theta)\right\}du$$

By changing variables we see that Ψ_k depends on $\tau-t$. This completes the proof. □

The forward price $F(t,\tau)$ gives us the joint dynamics of forward prices on each of the spot commodities. Hence, it is a d-variate process, giving the cross-commodity forward price dynamics. Recall that $\cdot$ denotes the pointwise product, and that we use the notation for the exponential function interchangeably, in the sense that $\exp(x)$ means elementwise exponentiation as long as x is a vector, and the matrix exponential when x is a matrix.

Note that since $\mathcal{C}_j(0)=0$ and $\Psi_k(0)=1$ for $k=1,\ldots,d$, it is easily seen that the expression for $F(t,\tau)$ is equal to $S(t)$ when $\tau=t$. This shows that the forward price converges to the spot at maturity, which it should by definition of the forward price as the conditional expectation of the spot at maturity. More interestingly is that the forward price dynamics is explicitly dependent on the stochastic volatility factors $Z_j(t)$. This has the interesting effect that even in the case of no spike components in the spot dynamics (*i.e.*, when $m=0$), the forward price dynamics will have jumps. That is, continuous spot price dynamics with stochastic volatility will imply forward price dynamics which jumps according to the jumps in the stochastic volatility.

We state the dynamics of the forward price.

Proposition 3.5. *Suppose the conditions in Prop. 3.4 holds. Then the dynamics of $F_k(t,\tau)$ of commodity k with respect to Q is*

$$\begin{aligned}\frac{dF_k(t,\tau)}{F_k(t-,\tau)}&=e_k^T\boldsymbol{e}^{A(\tau-t)}\Sigma^{1/2}(t)\,d\widehat{W}(t)\\&+\sum_{i=1}^m\int_{\mathbb{S}_d^+\backslash\{0\}}\left\{\exp\left(\frac{1}{2}e_k^T\mathrm{diag}(\mathcal{C}_i(\tau-t)V)+e_k^T\boldsymbol{e}^{B_i(\tau-t)}\eta_i\mathrm{diag}(V)\right)-1\right\}\widetilde{N}_i^Q(dt,dV)\\&+\sum_{j=m+1}^n\int_{\mathbb{S}_d^+\backslash\{0\}}\left\{\exp\left(\frac{1}{2}e_k^T\mathrm{diag}(\mathcal{C}_j(\tau-t)V)\right)-1\right\}\widetilde{N}_j^Q(dt,dV)\,.\end{aligned}$$

Here, $\widetilde{N}_j^Q(dt, dV) = N_j(dt, dV) - \exp(tr(V\Theta_j))\nu_{\widetilde{L}_j}(dV)\,dt$ *and* N_j *is the Poisson random measure of* $\widetilde{L}_j$*, for* $j = 1, \dots, n$*.*

Proof. First, let us notice that by definition, the process $t \mapsto F_k(t, \tau)$ is a martingale for $t \leq \tau$. From Prop. 3.4, we have in a compact form

$$F_k(t,\tau) = \Lambda_k(\tau) \exp\left(e_k^T \mathbf{e}^{A(\tau-t)} X(t) + e_k^T \mathbf{e}^{B(\tau-t)} Y(t) + \frac{1}{2}\mathrm{diag}(\mathcal{C}(\tau-t)Z(t)) \right) G_k(\tau - t)\,,$$

where we have collected all non-random terms into G, being a vector in $\mathbb{R}^d$. Since $F_k(t, \tau)$ depends on $X(t), Y(t)$ and $Z(t)$, the dynamics of F_k will necessarily be expressible in terms of the Q-Wiener process $\widehat{W}$ and the compensated Possion random measures of $\widetilde{L}_j$ under Q. Hence, when using Itô's Formula for jump processes (see e.g. Shiryaev [26]), we only need to focus on terms involving $d\widehat{W}$ and $\widetilde{N}_j^Q(dt, dV)$. To do this, we note that the dynamics of $Y(t)$ can be written as

$$dY(t) = (\mu + BY(t))\,dt + \eta\, d(\mathrm{diag}(\widetilde{L}(u)))\,.$$

Moreover, since $\mathcal{C}(\tau - t)$ and diag are linear operators on matrices, we have that F_k is a function of linear combinations of $Z_{u,v}(t)$ and $Y_u(t)$, for $u, v = 1, \dots, d$. Hence, the dynamics will consist of linear combinations of the elements of the $\widetilde{L}(t)$-matrix. Applying Itô's Formula taking into account all these considerations yields the result. $\square$

We see that there is a Samuelson effect in the forward price dynamics. The volatility appearing in the $d\widehat{W}$-term of the dynamics takes the form $e_k^T \exp(A(\tau - t))\Sigma^{1/2}(t)$. The contribution from $e_k^T \exp(A(\tau - t))$ is an "exponential scaling" of the stochastic spot volatility $\Sigma^{1/2}(t)$. Moreover, as time to maturity goes to zero, we obtain a convergence of the forward volatility to the spot volatility,

$$\lim_{\tau \downarrow t} e_k^T \mathbf{e}^{A(\tau-t)} \Sigma^{1/2}(t) = \Sigma^{1/2}(t)\,.$$

This yields a generalization of the Samuelson effect known in the one-dimensional case to cross-commodity forward prices. We remark that the one-dimensional Samuelson effect gives a forward volatility which is exponential dampening (in 'time-to-maturity') of the spot volatility. However, in the multi-dimensional case, the shape of $e_k^T \exp(A(\tau - t))\Sigma^{1/2}$ will be much richer than simply exponential decay in time to maturity towards spot volatility. In fact, one may get situations where the forward volatility is increasing rather than decreasing with time to maturity. For example, choosing A to be a matrix of CARMA-type (see Benth *et al.* [6]), we may get this situation, which is in contrast to the classical Samuelson effect. Observe that also the jump-terms in the dynamics of the forward price contributes with a Samuleson effect, however, this is much more complex to analyse.

In the next Proposition we show that the forward price will behave like the seasonal function in the long end of the market. To prove this result, we dispense with the restriction that the forward price is only defined up to maturities $\widetilde{T} < \infty$, but do an asymptotic consideration of F only focusing on the expression in Prop. 3.4.

Proposition 3.6. *Let $F(t,\tau)$ be given as in Prop. 3.4 and suppose $\lim_{t\to\infty} \ln \Psi(t)$ exists. Then,*

$$\lim_{\tau\to\infty}(\ln F(t,\tau) - \ln \Lambda(\tau)) = A^{-1}\theta_0 + \sum_{i=1}^{m} B_i^{-1}\mu_i + \lim_{\tau\to\infty} \ln \Psi(\tau)\,.$$

Here we understand the operations of the function $\ln$ *coordinate-wise.*

Proof. This result follows immediately from the assumption that the real parts of the eigenvalues of the matrices A, B_i and C_j are all negative, $i = 1,\dots,m$ and $j = 1,\dots,n$. $\square$

Note that the condition that $\Psi(\tau)$ has a limit is equivalent to the existence of a stationary dynamics of $\int_0^t \mathcal{C}_j(t-s)d\widetilde{L}_j(s)$ and $\int_0^t \mathbf{e}^{B_i(t-s)}\eta_i dL_i(u)$ under Q. If this is the case, then we can interpret $\lim_{\tau\to\infty} \ln \Psi(\tau)$ as the long-term mean value of the market price of risk.

From Prop. 3.6 above, contracts with maturities in the long end of the market will have forward prices which are basically equal to the seasonality function, adjusted by the stationary mean values of Y_i and Z_j and the market prices of risk, that is

$$F(t,\tau) \sim \text{const.} \cdot \Lambda(\tau)\,.$$

As a result of mean reversion of the spot prices, the forward prices are not reacting to changes in the spot in the long end but only following the seasonal mean adjusted by the market prices of risk.

3.3 Shapes of the forward curve

Note that we can view the forward price dynamics as a regression on the spot price, leverage terms and the volatility processes. Introducing the shorthand notation $\Theta(t,\tau) \in \mathbb{R}^d$ given by

$$\ln \Theta(t,\tau) \triangleq \ln \Psi(\tau - t) + \ln \Lambda(\tau) + A^{-1}(I - \mathbf{e}^{A(\tau-t)})\theta_0 + \sum_{i=1}^{m} B_i^{-1}(I - \mathbf{e}^{B_i(\tau-t)})\mu_i\,. \quad (3.14)$$

Then, from Prop. 3.4,

$$\begin{aligned}\ln F(t,\tau) = \ln \Theta(t,\tau) + \mathbf{e}^{A(\tau-t)} \ln S(t) + \sum_{i=1}^{m} \left(\mathbf{e}^{B_i(\tau-t)} - \mathbf{e}^{A(\tau-t)}\right) Y_i(t) \\ + \frac{1}{2}\text{diag}\{\sum_{j=1}^{n} \mathcal{C}_j(\tau-t)Z_j(t))\}\,. \quad (3.15)\end{aligned}$$

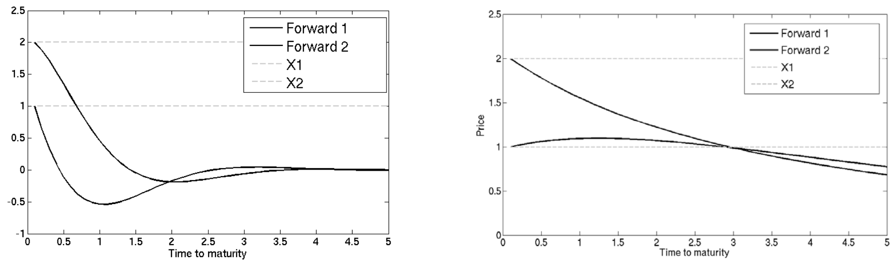
Here, Θ is a level adjustment function. The impact of the various factors on the forward price $F(t,\tau)$ goes through the matrix exponentials. In fact, the forward price of one commodity depends on the normal variation processes X, spike processes Y_i and volatility processes Z_j of all the commodities modelled. Hence, for example, if one of the commodities has a spike, then the forward prices of all the other commodities will be influenced. There is also a direct influence from the volaltility processes between the forwards both directly, and indirectly via the stochastic volatility $\Sigma(t)$ in the dynamics of X.

As noted in Andresen *et al.* [1], the mean-reverting structure represented by a matrix exponential has a richer structure than in the one-dimensional case, and we may include hump structures in the forward curve. We discuss the potential shapes of $\tau \mapsto F(t,\tau)$ in more detail. Since $A \in GL_d(\mathbb{R})$, it is diagonalizable. So it holds that $\mathbf{e}^{A(\tau-t)} = U\mathbf{e}^{\Lambda(\tau-t)}U^{-1}$, where U is a basis of eigenvectors and Λ is matrix with the eigenvalues of A on the diagonal and zero elsewhere (see *e.g.* Horn and Johnson [14]). Hence, an entry of the vector $\mathbf{e}^{A(\tau-t)}X(t)$ can be represented as

$$\sum_{i=1}^{d} a_{1i}\mathbf{e}^{\lambda_i(\tau-t)}X_1(t) + \sum_{i=1}^{d} a_{2i}\mathbf{e}^{\lambda_i(\tau-t)}X_2(t) + \ldots + \sum_{i=1}^{d} a_{di}\mathbf{e}^{\lambda_i(\tau-t)}X_d(t)\,,$$

for some constants $a_{ij} \in \mathbb{R}$ and eigenvalues λ_i, $i,j = 1,\ldots,d$. Consider first the Schwartz model with constant volatility, *i.e.* no contribution of the processes Y_i and Z_j in the forward price. If X is positive in all its components, λ_i is real and $a_{ij} \in \mathbb{R}^+$ for all $i,j = 1,\ldots,d$, then the forward is in backwardation since the eigenvalues have negative real-parts. The opposite conclusion (*i.e.* forward prices in contango) can be taken when X is negative in all its components. A more realistic situation with this model is the case where there are humps in the forward curve and where the forward is changing between backwardation and contango over time. This behavior has been observed for real market prices. For example, on page 216 in Geman [12] the forward curve of WTI oil is plotted together with the spot price. The shape of the curve varies over time from contango to backwardation, including positive humps in the short end. When the constants a_{ji} for fixed j are not all of the same sign and the entries of X have all a positive sign, then an entry of $\mathbf{e}^{A(\tau-t)}X(t)$ is given by a linear combination of increasing and decreasing exponentials which rise and decay at different speeds. Due to this the forward may alternate between backwardation and contango and humps may appear (see figure 3.3 (b)). Another possibility is the case of complex eigenvalues. This leads to an oscillating structures in the forward curve. So a change upward in the i-th component of X may cause a rise or fall of the forward depending on the time to maturity (see figure 3.3 (a)).

A similar analysis can be made for the spike processes Y_i. However, since Y_i is a pure jump process it will contribute to sudden changes in the forward curve. These humps may be upward or downward pointing depending on the time to maturity. The jumps caused by the spike process Y_i may be averaged out by jumps in the volatility process Z_i. The processes Y_i and Z_i are driven



(a) (b)

Figure 4.1: Paths of $e^{A(\tau-t)}X$ for (a) complex eigenvalues, (b) real eigenvalues of A, moreover $X = (1,\ 2)^T$ is taken constant.

by related subordinators L_i and $\widetilde{L}_i$. Hence Y_i and Z_i may have simultaneous jumps, however depending on the value of the matrices A, B_i and C_j an upward jump caused by the volatility process Z_i may simultaneously have a downward jump caused by the spike process Y_i. Hence the jumps may average out. Conversely, depending on the parameters A, B_i and C_j, the jumps in Y_i and Z_i may enlarge each other and lead to a big jump in the forward curve. This is a result of the inverse leverage effect in the spot model, which has a "double" impact on forward prices.

4 Transform-based pricing of options

Spread options are popular derivatives in the energy market to hedge price differences. For instance, spread options are traded on the difference in electricity forward prices in neighboring markets, or on the difference between electricity and one of its fuels including spark (electricity and gas) and dark (electricity and coal) spreads. On New York Mercantile Exchange (NYMEX) options on spreads between forwards on different refined oils are traded.

In this section we will consider pricing of options on a combination of forwards, with the spread as a special case. The dynamics of the forward is given by our multivariate model, which allows for the application of the Fourier method to pricing.

Consider an option written on a combination of the forwards expressed via the payoff function $p : \mathbb{R}^d \mapsto \mathbb{R}$. At exercise time $T \leq \tau$, the option pays out $p(F(T,\tau))$, with the forwards maturing at time $\tau \geq T$. Supposing that $p(F(T,\tau))$ is integrable with respect to the pricing measure Q defined in the previous section, we have that the option price at time $t \leq T$ becomes

$$C(t) = \mathbf{e}^{-r(T-t)}\mathbb{E}_Q\left[p(F(T,\tau)) \,|\, \mathcal{F}_t\right] , \tag{4.1}$$

where the constant $r > 0$ is the risk-free interest rate. As it turns out, the forward price, or rather its logarithm, has a semi-analytic cumulant function, which opens for applying the Fourier method to option pricing (see Carr and Madan [8] for a general treatment of Fourier methods in pricing of options). We now discuss this in more detail.

First, define the function

$$g(x) \triangleq p(\mathbf{e}^x)\,, \tag{4.2}$$

and observe that

$$g(\ln x) = p(x)\,,$$

where we used pointwise exponentials and logarithms. Suppose $g \in L^1(\mathbb{R}^d)$, the space of integrable functions on $\mathbb{R}^d$, and recall the d-dimensional Fourier transform as

$$\widehat{g}(y) = \int_{\mathbb{R}^d} g(x)\mathbf{e}^{-\mathrm{i}\langle x,y\rangle}\,dx\,. \tag{4.3}$$

If $\widehat{g} \in L^1(\mathbb{R}^d)$, then the inverse Fourier transform becomes

$$g(x) = \frac{1}{2\pi}\int_{\mathbb{R}^d} \widehat{g}(y)\mathbf{e}^{\mathrm{i}\langle y,x\rangle}\,dy\,. \tag{4.4}$$

See Folland [11] for these definitions. To price options, let us introduce the conditional cumulant function of the log-forward prices under Q: for $s \le t \le \tau$ and $x \in \mathbb{R}^d$, define

$$\widetilde{\phi}^{(s,t,\tau)}_{\ln F}(x) \triangleq \ln \mathbb{E}_Q\left[\mathbf{e}^{\mathrm{i}\langle x,\ln F(t,\tau)\rangle}\,|\,\mathcal{F}_t\right] \tag{4.5}$$

The following pricing relation holds:

Proposition 4.1. *Suppose that $g, \widehat{g} \in L^1(\mathbb{R}^d)$, where g is defined in* (4.2)*. Then*

$$C(t) = \boldsymbol{e}^{-r(T-t)}\frac{1}{2\pi}\int_{\mathbb{R}^d}\widehat{g}(y)\,\exp\left(\widetilde{\phi}^{(t,T,\tau)}_{\ln F}(y)\right)\,dy\,,$$

where $\widetilde{\phi}^{(t,T,\tau)}_{\ln F}(y)$, $t \le T \le \tau$ is the conditional characteristic function of $\ln F(T,\tau)$ defined in (4.5).

Proof. Since $g \in L^1(\mathbb{R}^d)$, using dominated convergence to commute integration and expectation (see Folland [11]), we conclude

$$\begin{aligned}
C(t) &= \mathbf{e}^{-r(T-t)}\mathbb{E}_Q[p(F(T,\tau))|\mathcal{F}_t] \\
&= \mathbf{e}^{-r(T-t)}\mathbb{E}_Q[g(\ln F(T,\tau))|\mathcal{F}_t] \\
&= \mathbf{e}^{-r(T-t)}\mathbb{E}_Q\left[\frac{1}{2\pi}\int_{\mathbb{R}^d}\widehat{g}(y)\mathbf{e}^{\mathrm{i}\langle y,\ln F(T,\tau)\rangle}\,dy\,|\mathcal{F}_t\right] \\
&= \mathbf{e}^{-r(T-t)}\frac{1}{2\pi}\int_{\mathbb{R}^d}\widehat{g}(y)\mathbb{E}_Q\left[\mathbf{e}^{\mathrm{i}\langle y,\ln F(T,\tau)\rangle}|\mathcal{F}_t\right]\,dy
\end{aligned}$$

$$= \mathbf{e}^{-r(T-t)} \frac{1}{2\pi} \int_{\mathbb{R}^d} \widehat{g}(y) \, \exp\left(\widetilde{\phi}_{\ln F}^{(t,T,\tau)}(y)\right) \, dy \, .$$

This proves the result. □

The two main ingredients in the pricing using Fourier methods are the transform of the payoff function, $\widehat{g}$ and the cumulant of the forward price under the pricing measure Q. We state a semi-analytical expression for the latter.

Proposition 4.2. *Assume the conditions of Prop. 3.4 hold. Then the conditional cumulant function of* $\ln F(t,\tau)$ *for* $s \le t \le \tau$ *defined in* (4.5) *is*

$$\widetilde{\phi}_{\ln F}^{(s,t,\tau)}(x) = \mathrm{i}x^T H(s,t,\tau) + \mathrm{i}x^T \boldsymbol{e}^{A(\tau-s)} X(s) + \sum_{i=1}^m \mathrm{i}x^T \boldsymbol{e}^{B_i(\tau-s)} Y_i(s)$$
$$+ \frac{1}{2}\sum_{j=1}^n \mathrm{i}x^T \mathrm{diag}\left(\mathcal{D}_j(t-s,\tau-t) Z_j(s)\right) - \frac{1}{2}\sum_{j=1}^n x^T \mathcal{C}_j(\tau-s) Z_j(s) x + \Xi(s,t,\tau,x) \, ,$$

for $x \in \mathbb{R}^d$*, where*

$$H(s,t,\tau) = \ln \Lambda(\tau) + \ln \Psi(\tau-t) + A^{-1}\left(I - \boldsymbol{e}^{A(\tau-t)}\right)\theta_0 + \sum_{i=1}^m B_i^{-1}\left(I - \boldsymbol{e}^{B_i(\tau-t)}\right)\mu_i$$
$$+ A^{-1}\left(\boldsymbol{e}^{A(\tau-t)} - \boldsymbol{e}^{A(\tau-s)}\right)\theta_0 + \sum_{i=1}^m B_i^{-1}\left(\boldsymbol{e}^{B_i(\tau-t)} - \boldsymbol{e}^{B_i(\tau-s)}\right)\mu_i$$

and

$$\begin{aligned}
&\Xi(s,t,\tau,x) \\
&\quad = \sum_{j=1}^m \int_0^{t-s} \Big\{ \phi_{\widetilde{L}_j}\left(\frac{1}{2}\mathrm{i}\mathcal{C}_j^*(\tau-t+v)(xx^T) + \frac{1}{2}\mathcal{D}_j^*(v,\tau-t)(J_d(x)) + J_d(x^T \boldsymbol{e}^{B_j(\tau-t+v)}\eta_j) - \mathrm{i}\Theta_j \right) \\
&\qquad\qquad - \phi_{\widetilde{L}_j}(-\mathrm{i}\Theta_j) \Big\} \, dv \\
&\quad + \sum_{j=m+1}^n \int_0^{t-s} \Big\{ \phi_{\widetilde{L}_j}\left(\frac{1}{2}\mathrm{i}\mathcal{C}_j^*(\tau-t+v)(xx^T) + \frac{1}{2}\mathcal{D}_j^*(v,\tau-t)(J_d(x)) - \mathrm{i}\Theta_j \right) - \phi_{\widetilde{L}_j}(-\mathrm{i}\Theta_j) \Big\} \, dv \, .
\end{aligned}$$

The family of linear operators $\mathcal{D}_j(u,v)$*,* $(u,v) \in \mathbb{R}_+^2$*, are defined as*

$$\mathcal{D}_j(u,v)X = \mathcal{C}_j(v)\boldsymbol{e}^{C_j u} X \boldsymbol{e}^{C_j^T u} \, ,$$

for $j = 1, \ldots, n$ *and a matrix* $X \in M_d(\mathbb{R})$.

Proof. From Prop. 3.4, it holds

$$\ln F(t,\tau) = \ln\Theta(t,\tau) + \mathbf{e}^{A(\tau)}X(t) + \sum_{i=1}^{m}\mathbf{e}^{B_i(\tau-t)}Y_i(t) + \frac{1}{2}\mathrm{diag}\{\sum_{j=1}^{n}\mathcal{C}_j(\tau-t)Z_j(t)\}\,,$$

where we recall the short-hand notation for $\Theta(t,\tau)$ defined in (3.14). Now, from the explicit solutions of the factors in (2.7), (2.8) and (2.9), together with the Girsanov change of the Brownian motion W, we find by adaptedness that

$$\begin{aligned}
\widetilde{\phi}_{\ln F}^{(s,t,\tau)}(x) &= \mathrm{i}x^T H(s,t,\tau) + \mathrm{i}x^T\mathbf{e}^{A(\tau-s)}X(s) + \mathrm{i}x^T\sum_{i=1}^{m}\mathbf{e}^{B_i(\tau-s)}Y_i(s) \\
&\quad + \frac{1}{2}\mathrm{i}x^T\mathrm{diag}\{\sum_{j=1}^{n}\mathcal{D}_j(t-s,\tau-t)Z_j(s)\} \\
&\quad + \ln\mathbb{E}_Q\Bigg[\exp\Bigg(\mathrm{i}x^T\int_s^t\mathbf{e}^{A(\tau-u)}\Sigma^{1/2}(u)\,d\widehat{W}(u) + \mathrm{i}x^T\sum_{i=1}^{m}\int_s^t\mathbf{e}^{B_i(\tau-u)}\eta_i\,dL_i(u) \\
&\qquad + \frac{1}{2}\mathrm{i}x^T\mathrm{diag}\{\sum_{j=1}^{n}\mathcal{C}_j(\tau-t)\int_s^t\mathbf{e}^{C_j(t-u)}\,d\widetilde{L}_j(u)\mathbf{e}^{C_j^T(t-u)}\}\Bigg)\mid\mathcal{F}_s\Bigg]
\end{aligned}$$

Define $\psi(s,t,\tau)$ as the logarithm of the conditional expectation in the expression above. Letting $\mathcal{G}_{s,t}$ be the σ-algebra generated by $\mathcal{F}_s$ and the paths of $\widetilde{L}_j(u)$, $s\leq u\leq t$, we find after using the tower property of the conditional expectation operator and the Gaussianity of Itô integrals of deterministic functions

$$\begin{aligned}
\psi(s,t,\tau) &= \ln\mathbb{E}_Q\Bigg[\mathbb{E}_Q\left[\exp\left(\mathrm{i}x^T\int_s^t\mathbf{e}^{A(\tau-u)}\Sigma^{1/2}(u)\,d\widehat{W}(u)\right)\mid\mathcal{G}_{s,t}^{\widetilde{L}}\right] \\
&\qquad\cdot\exp\left(\mathrm{i}x^T\sum_{i=1}^{m}\int_s^t\mathbf{e}^{B_i(\tau-u)}\eta_i\,dL_i(u)\right) \\
&\qquad\cdot\exp\left(\frac{1}{2}\mathrm{i}x^T\mathrm{diag}\{\sum_{j=1}^{n}\mathcal{C}_j(\tau-t)\int_s^t\mathbf{e}^{C_j(t-u)}\,d\widetilde{L}_j(u)\mathbf{e}^{C_j^T(t-u)}\}\right)\mid\mathcal{F}_t\Bigg] \\
&= \ln\mathbb{E}_Q\Bigg[\exp\left(-\frac{1}{2}x^T\mathrm{i}x^T\int_s^t\mathbf{e}^{A(\tau-u)}\Sigma(u)\mathbf{e}^{A^T(\tau-u)}x\right) \\
&\qquad\cdot\exp\left(\mathrm{i}x^T\sum_{i=1}^{m}\int_s^t\mathbf{e}^{B_i(\tau-u)}\eta_i\,dL_i(u)\right) \\
&\qquad\cdot\exp\left(\frac{1}{2}\mathrm{i}x^T\mathrm{diag}\{\sum_{j=1}^{n}\mathcal{C}_j(\tau-t)\int_s^t\mathbf{e}^{C_j(t-u)}\,d\widetilde{L}_j(u)\mathbf{e}^{C_j^T(t-u)}\}\right)\mid\mathcal{F}_t\Bigg]
\end{aligned}$$

Inspecting the proof of Lemma 3.3, we find

$$\int_s^t \mathbf{e}^{A(\tau-u)}\Sigma(u)\mathbf{e}^{A^T(\tau-u)}\,du = \sum_{j=1}^n \mathcal{C}_j(\tau-s)Z_j(s) + \int_s^t \mathcal{C}_j(\tau-u)\,d\widetilde{L}_j(u)\,.$$

By $\mathcal{F}_s$-adaptedness and independent increment property of Lévy processes, it holds

$$\begin{aligned}\psi(s,t,\tau) = &-\frac{1}{2}\sum_{j=1}^n x^T\mathcal{C}_j(\tau-s)Z_j(s)x\\ &+\ln\mathbb{E}_Q\Bigg[\exp\Bigg(-\frac{1}{2}\sum_{j=1}^n x^T\int_s^t \mathcal{C}_j(\tau-u)\,d\widetilde{L}_j(u)x\\ &\quad+\frac{1}{2}\mathrm{i}x^T\mathrm{diag}\{\sum_{j=1}^n\mathcal{C}_j(\tau-t)\int_s^t \mathbf{e}^{C_j(t-u)}\,d\widetilde{L}_j(u)\mathbf{e}^{C_j^T(t-u)}\}\\ &\quad+\sum_{i=1}^m \mathrm{i}x^T\int_s^t \mathbf{e}^{B_i(\tau-u)}\eta_i\,dL_i(u)\Bigg)\Bigg]\end{aligned}$$

We focus next on the last term, the logarithm of the expectation, which we denote by $\widetilde{\psi}(s,t,\tau)$. Observe first that

$$\mathcal{C}_j(\tau-t)\int_s^t \mathbf{e}^{C_j(t-u)}\,d\widetilde{L}_j(u)\mathbf{e}^{C_j^T(t-u)} = \int_s^t \mathcal{D}_j(t-u,\tau-t)\,d\widetilde{L}_j(u)\,.$$

But since for a matrix $A\in M_d(\mathbb{R})$, $x^T\mathrm{diag}(A) = tr\{J_d(x)A\}$,

$$\frac{1}{2}x^T\mathrm{diag}\{\sum_{j=1}^n\int_s^t \mathcal{D}_j(t-u,\tau-t)\,d\widetilde{L}_j(u)\} = \sum_{j=1}^n tr\{\frac{1}{2}J_d(x)\int_s^t \mathcal{D}_j(t-u,\tau-t)\,d\widetilde{L}_j(u)\}\,.$$

Furthermore, it holds that

$$x^T\int_s^t \mathbf{e}^{B_i(\tau-u)}\eta_i\,dL_i(u) = \int_s^t x^T\mathbf{e}^{B_i(\tau-u)}\eta_i\,dL_i(u) = tr\{\int_s^t J_d(x^T\mathbf{e}^{B_i(\tau-u)}\,d\widetilde{L}_i(u)\}\,,$$

and

$$-\frac{1}{2}x^T\int_s^t \mathcal{C}_j(\tau-u)\,d\widetilde{L}_j(u)x = \mathrm{i}tr\{\frac{1}{2}\mathrm{i}xx^T\int_s^t \mathcal{C}_j(\tau-u)\,d\widetilde{L}_j(u)\}\,.$$

Hence, collecting terms and using that $\widetilde{L}_j$ are independent for $j=1,\ldots,n$, we find

$$\widetilde{\psi}(s,t,\tau) = \sum_{j=1}^m \ln\mathbb{E}_Q\Bigg[\exp\Bigg(\mathrm{i}tr\Bigg\{\frac{1}{2}\mathrm{i}xx^T\int_s^t \mathcal{C}_j(\tau-u)\,d\widetilde{L}_j(u)$$

$$
\begin{aligned}
&\qquad +\frac{1}{2}J_d(x)\int_s^t \mathcal{D}_j(t-u,\tau-t)\,d\widetilde{L}_j(u)+\int_s^t J_d(x^T\mathbf{e}^{B_j(\tau-u)}\eta_j)\,d\widetilde{L}_j(u)\Big\}\Big)\Big]\\
&+\sum_{j=m+1}^{n}\ln\mathbb{E}_Q\Big[\exp\Big(\mathrm{i}tr\Big\{\frac{1}{2}\mathrm{i}xx^T\int_s^t \mathcal{C}_j(\tau-u)\,d\widetilde{L}_j(u)\\
&\qquad +\frac{1}{2}J_d(x)\int_s^t \mathcal{D}_j(t-u,\tau-t)\,d\widetilde{L}_j(u)\Big\}\Big)\Big]\\
&=\sum_{j=1}^{m}\int_s^t \phi_{\widetilde{L}_j}\left(\frac{1}{2}\mathrm{i}\mathcal{C}_j^*(\tau-u)(xx^T)+\frac{1}{2}\mathcal{D}_j^*(t-u,\tau-t)(J_d(x))+J_d(x^T\mathbf{e}^{B_j(\tau-u)}\eta_j)-\mathrm{i}\Theta_j\right)\\
&\qquad -\phi_{\widetilde{L}_j}(-\mathrm{i}\Theta_j)\,du\\
&\quad +\sum_{j=m+1}^{n}\int_s^t \phi_{\widetilde{L}_j}\left(\frac{1}{2}\mathrm{i}\mathcal{C}_j^*(\tau-u)(xx^T)+\frac{1}{2}\mathcal{D}_j^*(t-u,\tau-t)(J_d(x))-\mathrm{i}\Theta_j\right)\\
&\qquad -\phi_{\widetilde{L}_j}(-\mathrm{i}\Theta_j)\,du
\end{aligned}
$$

In the last equality, we used the same argumentation as is found in the proof of Prop. 3.4. After collecting terms, the proposition is proved. □

The fast Fourier transform (FFT) algorithm may be used to compute the option price efficiently, as long as we know the Fourier transform of the payoff function g. Note that implementing the FFT algorithm requires some numerical integration routines to evaluate the characteristic function of $\ln F$.

We consider the specific case of a call option on the spread between two forwards. The payoff function of such a contract is $p(x)=\max(x_1-x_2-K,0)$, where K is the strike price. Without loss of generality, we can suppose that $K=1$. The function g becomes

$$g(x)=\max(\mathbf{e}^{x_1}-\mathbf{e}^{x_2}-1,0)\,.$$

We observe that this function is not integrable on $\mathbb{R}^2$. However, following the idea in Carr and Madan [8], we can damp g by an exponential function. To this end, define for $\xi=(\xi_1,-\xi_2)$ with $\xi_1,\xi_2>0$,

$$g_\xi(x)=\mathbf{e}^{-\langle\xi,x\rangle}\max(\mathbf{e}^{x_1}-\mathbf{e}^{x_2}-1,0)\,. \tag{4.6}$$

We show that this becomes an integrable function under natural conditions on the damping factors ξ_1,ξ_2.

Lemma 4.3. *If $\xi_1-\xi_2>1$, then $g_\xi\in L^1(\mathbb{R}^2)$ where g_ξ is defined in* (4.6).

Proof. Note that the function g_ξ is non-zero whenever $x_1>\ln(\mathbf{e}^{x_2}+1)$. Thus, since $\xi_1>1$,

$$\int_{\mathbb{R}^2} g_\xi(x)\,dx=\int_{-\infty}^{\infty}\mathbf{e}^{\xi_2x_2}\int_{\ln(\mathbf{e}^{x_2}+1)}^{\infty}\mathbf{e}^{-\xi_1x_1}(\mathbf{e}^{x_1}-(\mathbf{e}^{x_2}+1))\,dx_1\,dx_2$$

$$= \frac{1}{\xi_1(\xi_1 - 1)} \int_{-\infty}^{\infty} \mathbf{e}^{\xi_2 x_2} (\mathbf{e}^{x_2} + 1)^{-(\xi_1 - 1)} \, dx_2 \, .$$

If $x_2 > 0$, we find that

$$\mathbf{e}^{\xi_2 x_2} (\mathbf{e}^{x_2} + 1)^{-(\xi_1 - 1)} = \mathbf{e}^{\xi_2 x_2} \mathbf{e}^{-(\xi_1 - 1) x_2} (1 + \mathbf{e}^{-x_2})^{-(\xi_1 - 1)} \leq \mathbf{e}^{(\xi_2 - \xi_1 + 1) x_2} \, .$$

By assumption $\xi_1 - \xi_2 > 1$, so $\xi_2 - \xi_1 + 1 < 0$. If $x_2 < 0$, then

$$\mathbf{e}^{\xi_2 x_2} (\mathbf{e}^{x_2} + 1)^{-(\xi_1 - 1)} \leq \mathbf{e}^{\xi_2 x_2} \, .$$

The Lemma follows. □

In the next Lemma we state the Fourier transform of g_ξ.

Lemma 4.4. *Suppose $\xi_2 > 0$ and $\xi_1 - \xi_2 > 1$. Then the Fourier transform of $g_\xi(x)$ defined in* (4.6) *is*

$$\widehat{g}_\xi(y) = \frac{\Gamma(\mathrm{i}(y_1 + y_2) - (1 + \xi_1 + \xi_2))\Gamma(-\mathrm{i}y_2 + \xi_2 + 2)}{\Gamma(\mathrm{i}y_1 + 1 - \xi_1)} \, , \tag{4.7}$$

where Γ denotes the gamma function.

Proof. For the proof we follow the approach of Hurd and Zhou [15] (Theorem 1). When one takes into account the exponential damping of the pay-off function g by $\mathbf{e}^{\langle \xi, x \rangle}$ then the above result follows. □

We have that

$$g(x) = \frac{1}{2\pi} \int_{\mathbb{R}^2} \widehat{g}_\xi(y) \mathbf{e}^{\mathrm{i}\langle (y - \mathrm{i}\xi), x \rangle} \, dy \, .$$

Thus, the price of a spread option entails in substituting y with $y - \mathrm{i}\xi$ in the formula for $C(t)$ in Prop. 4.1, and use $\widehat{g}_\xi$ instead of $\widehat{g}$. In addition comes an exponential integrability condition on $\ln F(T, \tau)$ in order to take into account the additional contribution from $\exp(\langle \xi, x \rangle)$.

An alternative approach to the Fourier method is to apply Monte Carlo simulation of the forward price dynamics $F(T, \tau)$ in the pricing of options. In practice, this means simulating matrix valued subordinators $\widetilde{L}_j$ and a multi-dimensional Wiener processes $\widehat{W}$ under Q. The latter can be simulated using classical sampling techniques. Efficient simulation methods for matrix-valued subordinators is in general an open question, however, for a specific class of such processes a method is proposed in Benth and Vos [7].

5 Conclusions

Based on the multivariate spot price model with Barndorff-Nielsen and Shephard stochastic volatility introduced in Benth and Vos [7], we derive the multivariate forward price dynamics. These analytical forward prices are calculated based on a combined Esscher-Girsanov change of measure where the risk premium is parametrized into a spike and volatility premium. Although the spot price has continuous sample paths in absence of a spike process, the implied forward curve will still exhibit jumps inherited from the stochastic volatility process. In the long end of the market the forward prices are basically equal to the seasonality function adjusted by the long-term means of the spike processes and volatility process and the market prices of risk. Since the mean-reverting structure of the involved matrix exponentials has a richer structure than in the one-dimensional case, the implied forward curve can alternate between backwardation and contango and humps may appear. Depending on the time to maturity a change in the spot can lead to various changes in the forward curve. We also discuss how a transform-based method can be used to price cross-commodity options on forwards. The particular case of spread options were analysed in more detail.

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Chapter 5

Futures pricing in electricity markets based on stable CARMA spot models

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